

# METASTABILITY OF THE TWO-DIMENSIONAL BLUME-CAPEL MODEL WITH ZERO CHEMICAL POTENTIAL AND SMALL MAGNETIC FIELD

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ABSTRACT. We consider the two-dimensional Blume-Capel model with zero chemical potential and small magnetic field evolving on a large but finite torus. We obtain sharp estimates for the transition time, we characterize the set of critical configurations, and we prove the metastable behavior of the dynamics as the temperature vanishes.

## 1. INTRODUCTION

Since its first rigorous mathematical treatment [20, 16, 10, 23, 9], metastability has been the subject of intensive investigation from different perspectives [13, 1, 22, 6, 14].

In [7, 8] Bovier, Eckhoff, Gayrard and Klein, BEGK from now on, have shown that the potential theory of Markov chains can outperform large deviations arguments and provides sharp estimates for several quantities appearing in metastability, such as the expectation of the exit time from a well or the probability to hit a configuration before returning to the starting configuration.

Developing further BEGK's potential-theoretic approach, and with an intensive use of data reduction through trace processes, Beltrán and one of the authors of this paper, BL from now on, devised a scheme to describe the evolution of a Markov chain among the wells, particularly effective when the dynamics presents several valleys of the same depth [2, 4, 5]. The outcome of the method can be understood as a model reduction through coarse-graining, or as the derivation of the evolution of the slow variables of the chain.

In the case of finite state Markov chains [3, 19], under minimal assumptions, BL's method permits the identification of all slow variables, the derivation of the time-scales at which they evolve and the characterization of their asymptotic dynamics.

In contrast with the pathwise approach [10, 23] and the transition path theory [13, 22], BEGK's and BL's approach do not attempt to describe the tube of typical trajectories in a transition between two valleys, nor does it identify the critical configurations which are visited with high probability in such transitions.

Nevertheless, under weak hypotheses, introduced in Section 3 below, potential-theoretic arguments together with data reduction through trace processes provide elementary identities and estimates which permit, without much effort, to characterize the critical configurations, and to compute the sub-exponential pre-factors of the expectation of hitting times. The purpose of this paper is to illustrate these assertions by examining the metastable behavior of the Blume-Capel model.

The Blume-Capel model is a two dimensional, nearest-neighbor spin system where the single spin variable takes three possible values:  $-1$ ,  $0$  and  $+1$ . One

can interpret it as a system of particles with spins. The value 0 of the spin at a lattice site corresponds to the absence of particles, whereas the values  $\pm 1$  correspond to the presence of a particle with the respective spin.

The metastability of the Blume-Capel model has been investigated by Cirillo and Olivieri [12], Manzo and Olivieri [21], and more recently by Cirillo and Nardi [11].

We consider here a Blume-Capel model with zero chemical potential and a small positive magnetic field. We examine its metastable behavior in the zero-temperature limit in a large, but fixed, two-dimensional square with periodic boundary conditions. In this case, there are two metastable states, the configurations where all spins are equal to  $-1$  or all spins equal to  $0$ , and one ground state, the configuration where all spins are equal to  $+1$ .

The main results state that starting from  $-1$ , the configuration where all spins are equal to  $-1$ , the chain visits  $0$  before hitting  $+1$ . We also characterize the set of critical configurations. These results are not new and appeared in [12, 11], but we present a proof which relies on a simple inequality from the potential theory of Markov chains. We compute the exact asymptotic values of the transition times, which corresponds to the life-time of the metastable states. The previous results on the transition time, based on the pathwise approach which relies on large deviations arguments, presented estimates with exponential errors. To complete the picture, we show that the expectation of the hitting time of the configuration  $0$  starting from  $-1$  is much larger than the transition time. This phenomenon, which may seem contradicting the fact that the chain visits  $0$  before hitting  $+1$ , occurs because the main contribution to the expectation comes from the event that the chain first hits  $+1$  and then visits  $0$ . The very small probability of this event is compensated by the very long time the chain remains at  $+1$ .

Finally, we prove the metastable behavior of the Blume-Capel model in the sense of BL. Let  $\Sigma$  be the set of configurations and let  $\mathcal{V}_a$  be a neighborhood of the configuration  $\mathbf{a}$ , where  $a = 0, \pm 1$ . For example  $\mathcal{V}_a = \{\mathbf{a}\}$ . Fix a real number  $\mathfrak{d} \notin \{-1, 0, +1\}$ , and let  $\phi : \Sigma \rightarrow \{-1, 0, 1, \mathfrak{d}\}$  be the projection defined by

$$\phi(\sigma) = \sum_{a=-1,0,1} a \mathbf{1}\{\sigma \in \mathcal{V}_a\} + \mathfrak{d} \mathbf{1}\left\{\sigma \notin \bigcup_{a=-1,0,+1} \mathcal{V}_a\right\}.$$

Denote the inverse of the temperature by  $\beta$ . We prove that there exists a time scale  $\theta_\beta$  for which  $\phi(\sigma(t\theta_\beta))$  converges, as  $\beta \rightarrow \infty$ , to a Markov chain in  $\{-1, 0, +1\}$ . The point  $+1$  is an absorbing point for this Markov chain, and the other jump rates are given by

$$r(-1, 0) = r(0, 1) = 1, \quad r(-1, 1) = r(0, -1) = 0.$$

As we said above this result can be interpreted as a model reduction by coarse-graining, or as the identification of a slow variable,  $\phi$ , whose evolution is asymptotically Markovian.

The article is divided as follows. In Section 2, we state the main results. In Section 3, we introduce the main tools used throughout the article and we present general results on finite-state reversible Markov chains. In section 4, we examine the transition from  $-1$  to  $0$ , and in Section 5 the one from  $0$  to  $+1$ . In Section 6, we analyze the hitting time of  $0$  starting from  $-1$ . In the last section, we prove the metastable behavior of the Blume-Capel model with zero chemical potential as the temperature vanishes.

## 2. NOTATION AND MAIN RESULTS

Fix  $L > 1$  and let  $\Lambda_L = \mathbb{T}_L \times \mathbb{T}_L$ , where  $\mathbb{T}_L = \{1, \dots, L\}$  is the discrete, one-dimensional torus of length  $L$ . Denote the configuration space by  $\Omega = \{-1, 0, 1\}^{\Lambda_L}$ , and by the Greek letters  $\sigma, \eta, \xi$  the configurations of  $\Omega$ . Hence,  $\sigma(x)$ ,  $x \in \Lambda_L$ , represents the spin at  $x$  of the configuration  $\sigma$ .

Fix an external field  $0 < h < 1$ , and denote by  $\mathbb{H} : \Omega \rightarrow \mathbb{R}$  the Hamiltonian given by

$$\mathbb{H}(\sigma) = \sum (\sigma(y) - \sigma(x))^2 - h \sum_{x \in \Lambda_L} \sigma(x), \quad (2.1)$$

where the first sum is carried over all unordered pairs of nearest-neighbor sites of  $\Lambda_L$ . Let  $n_0 = [2/h]$ , where  $[a]$  represents the integer part of  $a \in \mathbb{R}_+$ . We assume that  $L > n_0 + 3$ .

Denote by  $\beta > 0$  the inverse of the temperature and by  $\mu_\beta$  the Gibbs measure associated to the Hamiltonian  $\mathbb{H}$  at inverse temperature  $\beta$ ,

$$\mu_\beta(\sigma) = \frac{1}{Z_\beta} e^{-\beta \mathbb{H}(\sigma)}, \quad (2.2)$$

where  $Z_\beta$  is the partition function, the normalization constant which turns  $\mu_\beta$  a probability measure.

Denote by  $-\mathbf{1}, \mathbf{0}, +\mathbf{1}$  the configurations of  $\Omega$  with all spins equal to  $-1, 0, +1$ , respectively. These three configurations are local minima of the energy  $\mathbb{H}$ ,  $\mathbb{H}(+\mathbf{1}) < \mathbb{H}(\mathbf{0}) < \mathbb{H}(-\mathbf{1})$ , and  $+\mathbf{1}$  is the unique ground state.

The Blume-Capel dynamics is the continuous-time Markov chain on  $\Omega$ , denoted by  $\{\sigma_t : t \geq 0\}$ , whose infinitesimal generator  $L_\beta$  acts on functions  $f : \Omega \rightarrow \mathbb{R}$  as

$$\begin{aligned} (L_\beta f)(\sigma) &= \sum_{x \in \Lambda_L} R_\beta(\sigma, \sigma^{x,+}) [f(\sigma^{x,+}) - f(\sigma)] \\ &\quad + \sum_{x \in \Lambda_L} R_\beta(\sigma, \sigma^{x,-}) [f(\sigma^{x,-}) - f(\sigma)]. \end{aligned} \quad (2.3)$$

In this formula,  $\sigma^{x,\pm}$  represents the configuration obtained from  $\sigma$  by modifying the spin at  $x$  as follows,

$$\sigma^{x,\pm}(z) := \begin{cases} \sigma(x) \pm 1 \bmod 3 & \text{if } z = x \\ \sigma(z) & \text{if } z \neq x; \end{cases}$$

and the rates  $R_\beta$  are given by

$$R_\beta(\sigma, \sigma^{x,\pm}) = \exp \left\{ -\beta [\mathbb{H}(\sigma^{x,\pm}) - \mathbb{H}(\sigma)]_+ \right\}, \quad x \in \Lambda_L,$$

where  $a_+, a \in \mathbb{R}$ , stands for the positive part of  $a$ :  $a_+ = \max\{a, 0\}$ .

Clearly, the Gibbs measure  $\mu_\beta$  satisfies the detailed balance condition

$$\mu_\beta(\sigma) R_\beta(\sigma, \sigma^{x,\pm}) = \min \{ \mu_\beta(\sigma), \mu_\beta(\sigma^{x,\pm}) \} = \mu_\beta(\sigma^{x,\pm}) R_\beta(\sigma^{x,\pm}, \sigma),$$

and is therefore reversible for the dynamics.

Denote by  $D(\mathbb{R}_+, \Omega)$  the space of right-continuous functions  $\mathbf{x} : \mathbb{R}_+ \rightarrow \Omega$  with left-limits and by  $\mathbb{P}_\sigma = \mathbb{P}_\sigma^\beta$ ,  $\sigma \in \Omega$ , the probability measure on the path space  $D(\mathbb{R}_+, \Omega)$  induced by the Markov chain  $\sigma_t$  starting from  $\sigma$ . Expectation with respect to  $\mathbb{P}_\sigma$  is represented by  $\mathbb{E}_\sigma$ .

Denote by  $H_{\mathcal{A}}$ ,  $H_{\mathcal{A}}^+$ ,  $\mathcal{A} \subset \Omega$ , the hitting time and the time of the first return to  $\mathcal{A}$ , respectively:

$$H_{\mathcal{A}} = \inf \{t > 0 : \sigma_t \in \mathcal{A}\}, \quad H_{\mathcal{A}}^+ = \inf \{t > \tau_1 : \sigma_t \in \mathcal{A}\}, \quad (2.4)$$

where  $\tau_1$  represents the time of the first jump of the chain  $\sigma_t$ . We sometimes write  $H(\mathcal{A})$ ,  $H^+(\mathcal{A})$  instead of  $H_{\mathcal{A}}$ ,  $H_{\mathcal{A}}^+$ .

**Proposition 2.1.** *Starting from  $-1$  the chain visits the state  $0$  in its way to the ground state  $+1$ .*

$$\lim_{\beta \rightarrow \infty} \mathbb{P}_{-1}[H_{+1} < H_0] = 0.$$

Recall the definition of  $n_0$  introduced just below (2.1). Denote by  $\mathfrak{R}^l$  (resp.  $\mathfrak{R}_0^l$ ) the set of configurations in  $\{-1, 0, +1\}^{\Lambda_L}$  in which there are  $n_0(n_0 + 1) + 1$  spins which are not equal to  $-1$  (resp.  $0$ ). Of these spins,  $n_0(n_0 + 1)$  form a  $n_0 \times (n_0 + 1)$ -rectangle of  $0$  spins (resp.  $+1$  spins). The remaining spin not equal to  $-1$  is equal to  $0$  (resp.  $+1$ ) and is attached to the longest side of the rectangle (cf. configuration  $\sigma'$  in Figure 1 with  $n_0 = 5$ ). All configurations in  $\mathfrak{R}^l$  have the same energy, as well as all configurations in  $\mathfrak{R}_0^l$ .

The next result states that, starting from  $-1$ , the chain reaches the set  $\mathfrak{R}^l$  before hitting  $0$ .

**Proposition 2.2.** *We have that*

$$\lim_{\beta \rightarrow \infty} \mathbb{P}_{-1}[H_{\mathfrak{R}^l} < H_0] = 1, \quad \lim_{\beta \rightarrow \infty} \mathbb{P}_0[H_{\mathfrak{R}_0^l} < H_{+1}] = 1.$$

Denote by  $\lambda_{\beta}(\sigma)$ ,  $\sigma \in \Omega$ , the holding rates of the Markov chain  $\sigma_t$ , and by  $p_{\beta}(\eta, \xi)$ ,  $\eta, \xi \in \Omega$ , the jump probabilities, so that  $R_{\beta}(\eta, \xi) = \lambda_{\beta}(\eta)p_{\beta}(\eta, \xi)$ . Let  $M_{\beta}(\eta) = \mu_{\beta}(\eta)\lambda_{\beta}(\eta)$  be the stationary measure for the embedded discrete-time Markov chain.

Denote by  $\text{cap}(\mathcal{A}, \mathcal{B})$  the capacity between two disjoint subsets  $\mathcal{A}, \mathcal{B}$  of  $\Omega$ :

$$\text{cap}(\mathcal{A}, \mathcal{B}) = \text{cap}_{\beta}(\mathcal{A}, \mathcal{B}) = \sum_{\sigma \in \mathcal{A}} M_{\beta}(\sigma) \mathbb{P}_{\sigma}[H_{\mathcal{B}} < H_{\mathcal{A}}^+], \quad (2.5)$$

and let

$$\theta_{\beta} = \frac{\mu_{\beta}(-1)}{\text{cap}(-1, \{0, +1\})} \quad (2.6)$$

be the time-scale in which the Blume-Capel model reaches the ground state  $+1$  starting from the local minima  $-1$  or  $0$ .

**Proposition 2.3.** *For any configuration  $\eta \in \mathfrak{R}^l$  and any configuration  $\xi \in \mathfrak{R}_0^l$ ,*

$$\lim_{\beta \rightarrow \infty} \frac{\text{cap}(-1, \{0, +1\})}{\mu_{\beta}(\eta)} = \frac{4(2n_0 + 1)}{3} |\Lambda_L| = \lim_{\beta \rightarrow \infty} \frac{\text{cap}(0, \{-1, +1\})}{\mu_{\beta}(\xi)}.$$

The first identity of this proposition is proved in Section 4 and the second one in Section 5.

**Proposition 2.4.** *The expected time to visit the ground state starting from  $-1$  and from  $0$  are given by*

$$\lim_{\beta \rightarrow \infty} \frac{1}{\theta_{\beta}} \mathbb{E}_{-1}[H_{+1}] = 2, \quad \lim_{\beta \rightarrow \infty} \frac{1}{\theta_{\beta}} \mathbb{E}_0[H_{+1}] = 1.$$

We have seen in Proposition 2.1 that starting from  $-1$  the process reaches  $0$  before visiting  $+1$ . In contrast, the next identity shows that the main contribution to the expectation  $\mathbb{E}_{-1}[H_0]$  comes from the event in which the process, starting from  $-1$ , first visits  $+1$ , remains there for a very long time and then reaches  $0$ . We have that

$$\frac{1}{\theta_\beta} \mathbb{E}_{-1}[H_0] = (b + o(1)) \frac{\mu_\beta(+1)}{\mu_\beta(0)} \mathbb{P}_{-1}[H_{+1} < H_0], \quad (2.7)$$

where  $o(1)$  is an expression which vanishes as  $\beta \uparrow \infty$ , and

$$\lim_{\beta \rightarrow \infty} \frac{\mu_\beta(+1)}{\mu_\beta(0)} \mathbb{P}_{-1}[H_{+1} < H_0] = \infty. \quad (2.8)$$

A self-avoiding path  $\gamma$  from  $\mathcal{A}$  to  $\mathcal{B}$ ,  $\mathcal{A}, \mathcal{B} \subset \Omega$ ,  $\mathcal{A} \cap \mathcal{B} = \emptyset$ , is a sequence of configurations  $(\xi_0, \xi_1, \dots, \xi_n)$  such that  $\xi_0 \in \mathcal{A}$ ,  $\xi_n \in \mathcal{B}$ ,  $\xi_j \notin \mathcal{A} \cup \mathcal{B}$ ,  $0 < j < n$ ,  $\xi_i \neq \xi_j$ ,  $i \neq j$ ,  $R_\beta(\xi_i, \xi_{i+1}) > 0$ ,  $0 \leq i < n$ . Denote by  $\Gamma_{\mathcal{A}, \mathcal{B}}$  the set of self-avoiding paths from  $\mathcal{A}$  to  $\mathcal{B}$  and let

$$\mathbb{H}(\mathcal{A}, \mathcal{B}) := \min_{\gamma \in \Gamma_{\mathcal{A}, \mathcal{B}}} \mathbb{H}(\gamma), \quad \mathbb{H}(\gamma) := \max_{0 \leq i \leq n} \mathbb{H}(\xi_i). \quad (2.9)$$

Let  $\mathcal{M} = \{-1, 0, +1\}$  be the set of ground configurations of the main wells, and let  $\mathcal{V}_\eta$ ,  $\eta \in \mathcal{M}$ , be a neighborhood of the configuration  $\eta$ . We assume that all configurations  $\sigma \in \mathcal{V}_\eta$ ,  $\sigma \neq \eta$ , fulfill the conditions

$$\mathbb{H}(\sigma) > \mathbb{H}(\eta), \quad \mathbb{H}(\eta, \sigma) - \mathbb{H}(\eta) < \mathbb{H}(-1, \{0, +1\}) - \mathbb{H}(-1). \quad (2.10)$$

The right hand side in the second condition represents the energetic barrier the chain needs to surmount to reach the set  $\{0, +1\}$  starting from  $-1$ , while the left hand side represents the energetic barrier to go from  $\eta$  to  $\sigma$ .

It follows from Proposition 2.3 that

$$\mathbb{H}(-1, \{0, +1\}) - \mathbb{H}(-1) = \mathbb{H}(0, \{-1, +1\}) - \mathbb{H}(0). \quad (2.11)$$

We may therefore replace the expression on the right hand side of (2.10) by the one on the right hand side of the previous formula.

Clearly,  $\mathcal{V}_\eta = \{\eta\}$ ,  $\eta \in \mathcal{M}$ , is an example of neighborhoods satisfying (2.10). Let  $\mathcal{V}$  be the union of the three neighborhoods,  $\mathcal{V} = \cup_{\eta \in \mathcal{M}} \mathcal{V}_\eta$ , and let  $\pi : \mathcal{M} \rightarrow \{-1, 0, 1\}$  be the application which provides the magnetization of the states  $-1, 0, +1$ :  $\pi(-1) = -1$ ,  $\pi(0) = 0$ ,  $\pi(+1) = 1$ . Denote by  $\Psi = \Psi_{\mathcal{V}} : \Omega \rightarrow \{-1, 0, 1, [\beta]\}$  the projection defined by  $\Psi(\sigma) = \pi(\eta)$  if  $\sigma \in \mathcal{V}_\eta$ ,  $\Psi(\sigma) = [\beta]$ , otherwise:

$$\Psi(\sigma) = \sum_{\eta \in \mathcal{M}} \pi(\eta) \mathbf{1}\{\sigma \in \mathcal{V}_\eta\} + [\beta] \mathbf{1}\left\{\sigma \notin \bigcup_{\eta \in \mathcal{M}} \mathcal{V}_\eta\right\}.$$

Recall from [3] the definition of the soft topology.

**Theorem 2.5.** *The speeded-up, hidden Markov chain  $X_\beta(t) = \Psi(\sigma(\theta_\beta t))$  converges in the soft topology to the continuous-time Markov chain  $X(t)$  on  $\{-1, 0, 1\}$  in which 1 is an absorbing state, and whose jump rates are given by*

$$r(-1, 0) = r(0, 1) = 1, \quad r(-1, 1) = r(0, -1) = 0.$$

**Remark 2.6.** *Denote by  $\mathcal{B}_\eta$ ,  $\eta \in \mathcal{M}$ , the basin of attraction of  $\eta$ :*

$$\mathcal{B}_\eta = \left\{ \sigma : \lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma[H_{\mathcal{M} \setminus \{\eta\}} < H_\eta] = 0 \right\}.$$

*We prove in (7.11) that  $\mathcal{V}_\eta \subset \mathcal{B}_\eta$ .*

### 3. METASTABILITY OF REVERSIBLE MARKOV CHAINS

We present in this section some results on reversible Markov chains. Consider two nonnegative sequences  $(a_N : N \geq 1)$ ,  $(b_N : N \geq 1)$ . The notation  $a_N \prec b_N$  (resp.  $a_N \preceq b_N$ ) indicates that  $\limsup_{N \rightarrow \infty} a_N/b_N = 0$  (resp.  $\limsup_{N \rightarrow \infty} a_N/b_N < \infty$ ), while  $a_N \approx b_N$  means that  $a_N \preceq b_N$  and  $b_N \preceq a_N$ .

A set of nonnegative sequences  $(a_N^r : N \geq 1)$ ,  $r \in \mathfrak{R}$ , is said to be *ordered* if for all  $r \neq s \in \mathfrak{R}$   $\arctan(a_N^r/a_N^s)$  converges.

Fix a finite set  $E$ . Consider a sequence of continuous-time,  $E$ -valued Markov chains  $\{\eta_t^N : t \geq 0\}$ ,  $N \geq 1$ . We assume, throughout this section, that the chain  $\eta_t^N$  is *irreducible*, that the unique stationary state, denoted by  $\mu_N$ , is *reversible*, and that the jump rates of the chain  $\eta_t^N$ , denoted by  $R_N(x, y)$ ,  $x \neq y \in E$ , satisfy the following hypothesis. Let  $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ , let  $\mathbb{B}$  be the bonds of  $E$ :  $\mathbb{B} = \{(x, y) \in E \times E : x \neq y \in E\}$ , and let  $\mathfrak{A}_m$ ,  $m \geq 1$ , be the set of functions  $k : \mathbb{B} \rightarrow \mathbb{Z}_+$  such that  $\sum_{(x, y) \in \mathbb{B}} k(x, y) = m$ .

**Assumption 3.1.** *We assume that for every  $m \geq 1$  the set of sequences*

$$\left\{ \prod_{(x, y) \in \mathbb{B}} R_N(x, y)^{k(x, y)} : N \geq 1 \right\}, \quad k \in \mathfrak{A}_m$$

*is ordered.*

This assumption is slightly weaker than the hypotheses (2.1), (2.2) in [3], but strong enough to derive all results presented in that article. It is also not difficult to check that the Blume-Capel model introduced in the previous section fulfills Assumption 3.1.

We adopt here similar notation to the one introduced in the previous section. For example,  $\lambda_N(x)$  represents the holding rates of the Markov chain  $\eta_t^N$ ,  $p_N(x, y)$ ,  $x, y \in E$ , the jump probabilities, and  $M_N(x) = \mu_N(x)\lambda_N(x)$  a stationary measure of the embedded discrete-time Markov chain. Analogously,  $\mathbb{P}_x = \mathbb{P}_x^N$ ,  $x \in E$ , represents the distribution of the Markov chain  $\eta_t^N$  starting from  $x$  and  $\mathbb{E}_x$  the expectation with respect to  $\mathbb{P}_x$ .

Denote by  $H_A$  (resp.  $H_A^+$ ),  $A \subset E$ , the hitting time of (resp. the return time to) the set  $A$ , introduced in (2.4), and by  $\text{cap}(A, B)$  the capacity between two disjoint subsets  $A, B$  of  $E$ , as defined in (2.5).

The following identity will be used often. Let  $A, B$  be two disjoint subsets of  $E$  and let  $x$  be a point which does not belong to  $A \cup B$ . We claim that

$$\mathbb{P}_x[H_A < H_B] = \frac{\mathbb{P}_x[H_A < H_{B \cup \{x\}}^+]}{\mathbb{P}_x[H_{A \cup B} < H_x^+]}. \quad (3.1)$$

To prove this identity, intersect the event  $\{H_A < H_B\}$  with the set  $\{H_x^+ < H_{A \cup B}\}$  and its complement, and then apply the strong Markov property to get that

$$\mathbb{P}_x[H_A < H_B] = \mathbb{P}_x[H_x^+ < H_{A \cup B}] \mathbb{P}_x[H_A < H_B] + \mathbb{P}_x[H_A < H_{B \cup \{x\}}^+].$$

To obtain (3.1) it remains to subtract the first term on the right hand side from the left hand side.

Multiply and divide the right hand side of (3.1) by  $M(x)$  and recall the definition of the capacity to obtain that

$$\mathbb{P}_x[H_A < H_B] = \frac{M(x)\mathbb{P}_x[H_A < H_{B \cup \{x\}}^+]}{\text{cap}(x, A \cup B)} \leq \frac{M(x)\mathbb{P}_x[H_A < H_x^+]}{\text{cap}(x, A \cup B)}. \quad (3.2)$$

Hence, by definition of capacity and since, by [17, Lemma 2.2], the capacity is monotone.

$$\mathbb{P}_x[H_A < H_B] \leq \frac{\text{cap}(x, A)}{\text{cap}(x, A \cup B)} \leq \frac{\text{cap}(x, A)}{\text{cap}(x, B)}. \quad (3.3)$$

For any disjoint subsets  $A$  and  $B$  of  $E$ ,

$$\text{cap}(A, B) \approx \max_{x \in A} \max_{y \in B} \text{cap}(x, y). \quad (3.4)$$

Indeed, on the one hand, by monotonicity of the capacity,

$$\text{cap}(A, B) \geq \max_{x \in A} \text{cap}(x, B) \geq \max_{x \in A} \max_{y \in B} \text{cap}(x, y).$$

On the other hand, by definition of the capacity,

$$\text{cap}(A, B) = \sum_{y \in B} M(y) \mathbb{P}_y[H_A < H_B^+] \leq \sum_{x \in A} \sum_{y \in B} M(y) \mathbb{P}_y[H_x < H_B^+],$$

Therefore,

$$\text{cap}(A, B) \leq \sum_{x \in A} \text{cap}(x, B) \leq |A| \max_{x \in A} \text{cap}(x, B), \quad (3.5)$$

where  $|A|$  stands for the cardinality of  $A$ . Repeating this argument for  $B$  in place of  $A$ , we conclude the proof of (3.4).

Let  $G_N : E \times E \rightarrow \mathbb{R}_+$  be given by  $G_N(x, y) = \mu_N(x) R_N(x, y)$  and note that  $G_N$  is symmetric. In the electrical network interpretation of reversible Markov chains,  $G_N(x, y)$  represents the conductance of the bond  $(x, y)$ . Recall that a self-avoiding path  $\gamma$  from  $A$  to  $B$ ,  $A, B \subset E$ ,  $A \cap B = \emptyset$ , is a sequence of sites  $(x_0, x_1, \dots, x_n)$  such that  $x_0 \in A$ ,  $x_n \in B$ ,  $x_j \notin A \cup B$ ,  $0 < j < n$ ,  $x_i \neq x_j$ ,  $i \neq j$ ,  $R_N(x_i, x_{i+1}) > 0$ ,  $0 \leq i < n$ . Denote by  $\Gamma_{A,B}$  the set of self-avoiding paths from  $A$  to  $B$  and let

$$G_N(A, B) := \max_{\gamma \in \Gamma_{A,B}} G_N(\gamma), \quad G_N(\gamma) := \min_{0 \leq i < n} G_N(x_i, x_{i+1}).$$

By [3, Lemma 4.2], for every disjoint subsets  $A, B$  of  $E$ , the limit

$$\lim_{N \rightarrow \infty} \frac{\text{cap}(A, B)}{G_N(A, B)} \text{ exists and belongs to } (0, \infty). \quad (3.6)$$

**Remark 3.2.** Suppose that the stationary state  $\mu_N$  is a Gibbs measure associated to an energy  $\mathbb{H}$ , and that we are interested in the Metropolis dynamics:  $\mu_N(x) = Z_N^{-1} \exp\{-N\mathbb{H}(x)\}$ , where  $Z_N$  is the partition function,  $R_N(x, y) = \exp\{-N[\mathbb{H}(y) - \mathbb{H}(x)]_+\}$ . In this context,  $G_N(x, y) = Z_N^{-1} \exp\{-N \max\{\mathbb{H}(x), \mathbb{H}(y)\}\}$ . In particular, for a path  $\gamma = (x_0, x_1, \dots, x_n)$ ,  $G_N(\gamma) = Z_N^{-1} \exp\{-N \max_i \mathbb{H}(x_i)\}$ , and for two disjoint subsets  $A, B$  of  $E$ ,

$$\begin{aligned} G_N(A, B) &= \frac{1}{Z_N} \exp\left\{-N \min_{\gamma \in \Gamma_{A,B}} \max_{x \in \gamma} \mathbb{H}(x)\right\} \\ &= \frac{1}{Z_N} \exp\left\{-N \mathbb{H}(x_{A,B})\right\} = \mu_N(x_{A,B}), \end{aligned}$$

where  $x_{A,B}$  represents the configuration with highest energy in the optimal path joining  $A$  to  $B$ .

**Lemma 3.3.** *Let  $E_1$  be a subset of  $E$ . Assume that for every  $y \notin E_1$ ,  $z \in E_1$  such that  $\mu(z) \preceq \mu(y)$ ,*

$$\frac{\text{cap}(y, z)}{\mu(z)} \prec \frac{\text{cap}(y, E_1)}{\mu(y)}. \quad (3.7)$$

*Then, for any  $B \subset E_1$ ,  $x \in E_1 \setminus B$ ,*

$$\mathbb{E}_x[H_B] = (1 + o(1)) \frac{1}{\text{cap}(x, B)} \sum_{y \in E_1} \mu(y) \mathbb{P}_y[H_x < H_B].$$

*Proof.* By [2, Proposition 6.10],

$$\mathbb{E}_x[H_B] = \frac{1}{\text{cap}(x, B)} \sum_{y \in E} \mu(y) \mathbb{P}_y[H_x < H_B].$$

Denote by  $\mathbb{P}_z^1$ ,  $z \in E_1$ , the distribution of the trace of  $\sigma(t)$  on the set  $E_1$  starting from  $z$  (cf. [2, Section 6] for the definition of trace), and let  $q(y, z) = \mathbb{P}_y[H_{E_1} = H_z]$ ,  $y \notin E_1$ ,  $z \in E_1$ . Decomposing the previous sum according to  $y \in E_1$ ,  $y \notin E_1$ , since  $B$  and  $x$  are contained in  $E_1$ , we can write it as

$$\begin{aligned} & \sum_{y \in E_1} \mu(y) \mathbb{P}_y^1[H_x < H_B] + \sum_{y \notin E_1} \sum_{z \in E_1} \mu(y) q(y, z) \mathbb{P}_z^1[H_x < H_B] \\ &= \sum_{y \in E_1} \mu(y) \mathbb{P}_y^1[H_x < H_B] + \sum_{z \in E_1} \mathbb{P}_z^1[H_x < H_B] \sum_{y \notin E_1} \mu(y) q(y, z). \end{aligned} \quad (3.8)$$

We claim that for  $y \notin E_1$ ,  $z \in E_1$ ,

$$\mu(y) q(y, z) = \mu(y) \mathbb{P}_y[H_z < H_{E_1 \setminus \{z\}}] \prec \mu(z). \quad (3.9)$$

If  $\mu(y) \prec \mu(z)$ , there is nothing to prove. Assume that  $\mu(z) \preceq \mu(y)$ . In this case, by (3.3) and by (3.7), the second term in the previous expression is bounded by

$$\frac{\mu(y) \text{cap}(y, z)}{\text{cap}(y, E_1)} \prec \mu(z).$$

This proves claim (3.9) and that the second term in the last equation of (3.8) is of smaller order than the first, as asserted.  $\square$

**Remark 3.4.** *In Lemma 3.3, the set  $E_1$  has to be interpreted as the union of wells. In the set-up of the Metropolis dynamics introduced in Remark 3.2, by (3.6) and Remark 3.2, for two disjoint subsets  $A, B$  of  $E$ ,  $\text{cap}(A, B)/\mu_N(x_{A,B})$  converges, as  $N \uparrow \infty$ , to a real number in  $(0, \infty)$ . Hence, assumption (3.7) requires that for all  $z \in E_1$ ,  $y \notin E_1$  such that  $\mathbb{H}(y) \leq \mathbb{H}(z)$ ,*

$$\mathbb{H}(x_{y, E_1}) - \mathbb{H}(y) < \mathbb{H}(x_{y, z}) - \mathbb{H}(z). \quad (3.10)$$

*In other words, it requires the energy barrier from  $y$  to  $E_1$  to be smaller than the one from  $z$  to  $y$ .*

The condition (3.10) may seem unnatural, as one would expect on the right hand side  $\mathbb{H}(x_{z, \check{E}_z}) - \mathbb{H}(z)$  instead of  $\mathbb{H}(x_{y, z}) - \mathbb{H}(z)$ , where  $\check{E}_z$  represents the union of the wells which do not contain  $z$ . However, since in the applications the set  $E_1$  represents the union of wells, and since  $\mathbb{H}(y) \leq \mathbb{H}(z)$ , to reach  $y$  from  $z$  the chain has to jump from one well to another and therefore one should have  $\mathbb{H}(x_{z, \check{E}_z}) - \mathbb{H}(z) \leq \mathbb{H}(x_{y, z}) - \mathbb{H}(z)$ .



**Lemma 3.5.** *Fix two points  $a \neq b \in E$ . The set of sequences  $\mu_N(x)\mathbb{P}_x[H_a < H_b]$ ,  $x \in E \setminus \{b\}$ , is ordered.*

*Proof.* Fix two points  $x \neq y \in E \setminus \{b\}$ . We need to show that the ratio  $\mu_N(x)\mathbb{P}_x[H_a < H_b] / \mu_N(y)\mathbb{P}_y[H_a < H_b]$  either converges to some value in  $[0, \infty)$ , or increases to  $\infty$ .

Assume that  $x \neq a$ ,  $y \neq a$ , and consider the trace of the process  $\eta^N(t)$  on  $A = \{a, b, x, y\}$ . By [2, Section 6], the stationary measure of the trace is the measure  $\mu_N$  conditioned to  $A$ . Denote by  $\mathbb{P}_z^A$  the distribution of the trace starting from  $z$ . It is clear that  $\mathbb{P}_z[H_a < H_b] = \mathbb{P}_z^A[H_a < H_b]$ . Therefore,

$$\frac{\mu_N(x)\mathbb{P}_x[H_a < H_b]}{\mu_N(y)\mathbb{P}_y[H_a < H_b]} = \frac{\mu_N^A(x)\mathbb{P}_x^A[H_a < H_b]}{\mu_N^A(y)\mathbb{P}_y^A[H_a < H_b]},$$

where  $\mu_N^A$  represents the measure  $\mu_N$  conditioned to  $A$ . Since  $A$  has only four elements, it is not difficult to show that

$$\mathbb{P}_x^A[H_a < H_b] = \frac{p^A(x, a) + p^A(x, y)p^A(y, a)}{1 - p^A(x, y)p^A(y, x)},$$

where  $p^A(z, z')$  represents the jump probabilities of the trace process. In particular, multiplying the numerator and the denominator of the penultimate ratio by  $\lambda^A(x)\lambda^A(y)$ , where  $\lambda^A$  stands for the holding rates of the trace process, yields that the penultimate ratio is equal to

$$\frac{\mu_N^A(x)\{\lambda^A(y)R^A(x, a) + R^A(x, y)R^A(y, a)\}}{\mu_N^A(y)\{\lambda^A(x)R^A(y, a) + R^A(y, x)R^A(x, a)\}},$$

where  $R^A$  is the jump rates of the trace process. By reversibility of the trace process, this expression is equal to

$$\frac{\lambda^A(y)R^A(a, x) + R^A(y, x)R^A(a, y)}{\lambda^A(x)R^A(a, y) + R^A(x, y)R^A(a, x)}.$$

By [3, Lemma 4.3], the set of jump rates  $R^A(x, y)$  satisfies Assumption 3.1. Since  $\lambda^A(z) = \sum_{z' \in A, z' \neq z} R^A(z, z')$ , by Assumption 3.1, the previous expression either converges to some  $a \in [0, \infty)$ , or increases to  $+\infty$ . This completes the proof of the assertion in the case where  $x, y \notin \{a, b\}$ .

The case where  $x = a$  or  $y = a$  is simpler and left to the reader.  $\square$

#### 4. PROOFS OF PROPOSITIONS 2.1, 2.2 AND 2.3.A

We examine in this section the metastable behavior of the Blume-Capel model starting from  $-1$ . We first consider isovolumetric inequalities. Denote by  $\|\cdot\|$  the Euclidean norm of  $\mathbb{R}^2$ . A subset  $A$  of  $\mathbb{Z}^2$  is said to be connected if for every  $x, y \in A$ , there exists a path  $\gamma = (x = x_0, x_1, \dots, x_n = y)$  such that  $x_i \in A$ ,  $\|x_{i+1} - x_i\| = 1$ ,  $0 \leq i < n$ . Denote by  $\mathcal{C}_n$ ,  $n \geq 1$ , the class of connected subset of  $\mathbb{Z}^2$  with  $n$  points and by  $P(A)$  the perimeter of a set  $A \in \mathcal{C}_n$ :

$$P(A) = \#\{(x, y) \in \mathbb{Z}^2 : x \in A, y \notin A, \|x - y\| = 1\}.$$

where  $\#B$  stands for the cardinality of  $B$ .

**Assertion 4.A.** *For every  $A \in \mathcal{C}_n$ ,  $n \geq 1$ ,  $P(A) \geq 4\sqrt{n}$ .*

*Proof.* For  $A \in \mathcal{C}_n$ , denote by  $R$  the smallest rectangle which contains  $A$ , and by  $a \leq b$  the length of the sides of the rectangle  $R$ . Since  $A$  is connected, and since  $R$  is the smallest rectangle which contains  $A$ ,  $P(A) \geq 2(a+b) \geq 2 \min\{a+b : a, b \in \mathbb{R}, ab \geq n\} = 4\sqrt{n}$ .  $\square$

**Assertion 4.B.** *A set  $A \in \mathcal{C}_m$ ,  $m = n_0(n_0 + 1)$ , is either a  $n_0 \times (n_0 + 1)$  rectangle or has perimeter  $P(A) \geq 4(n_0 + 1)$ .*

*Proof.* Fix  $A \in \mathcal{C}_m$ , and recall the notation introduced in the proof of the previous assertion. We may restrict our study to the case where the length of the shortest side of  $R$ , denoted by  $a$ , is less than or equal to  $n_0$ , otherwise the perimeter is larger than or equal to  $4(n_0 + 1)$ . If  $a = n_0$ , either  $b = n_0 + 1$ , in which case, to match the volume,  $A$  must be a  $n_0 \times (n_0 + 1)$  rectangle, or  $b \geq n_0 + 2$ , in which case the perimeter is larger than or equal to  $4(n_0 + 1)$ . If  $a = n_0 - j$  for some  $j \geq 1$ , then  $b = n_0 + k$  for some  $k \geq 1$  because the volume has to be at least  $n_0^2$ . Actually, we need  $(n_0 + k)(n_0 - j) \geq n_0(n_0 + 1)$ , i.e.,  $(k - j)n_0 \geq n_0 + kj$ . This forces  $k - j \geq 2$  and, in consequence, the perimeter  $P \geq 4(n_0 + 1)$ .  $\square$

We may extend the definition of the energy  $\mathbb{H}$  introduced in (2.1) to configuration in  $\{-1, 0, 1\}^{\mathbb{Z}^2}$ . For such configurations, while  $\mathbb{H}(\sigma)$  is not well defined,  $\mathbb{H}(\sigma) - \mathbb{H}(-1)$  is well defined if  $\sigma_x = -1$  for all but a finite number of sites.

Denote by  $\partial_+ A$  the outer boundary of a connected finite subset  $A$  of  $\mathbb{Z}^2$ :  $\partial_+ A = \{x \notin A : \exists y \in A \text{ s.t. } \|y - x\| = 1\}$ .

**Assertion 4.C.** *Let  $A \in \mathcal{C}_n$ ,  $1 \leq n \leq (n_0 + 1)^2$ , and let  $\sigma$  be a configuration of  $\{-1, 0, 1\}^{\mathbb{Z}^2}$  whose spins in  $A$  are equal to  $+1$  and whose spins in  $\partial_+ A$  are either  $0$  or  $-1$ . Let  $\sigma^*$  be the configuration obtained from  $\sigma$  by switching all spins in  $A$  to  $0$ . Then,  $\mathbb{H}(\sigma) \geq \mathbb{H}(\sigma^*) + 2$ .*

*Proof.* By definition of the energy and since  $A$  has  $n$  points,  $\mathbb{H}(\sigma) - \mathbb{H}(\sigma^*) = -hn + P_0 + 3P_{-1} \geq -hn + P$ , where  $P_0$  (resp.  $P_{-1}$ ) represents the number of unordered pairs  $\{x, y\}$  such that  $x \in A$ ,  $y \in \partial_+ A$ ,  $\sigma_y = 0$  (resp.  $\sigma_y = -1$ ), and where  $P = P_0 + P_{-1}$  is the perimeter of the set  $A$ .

It remains to show that  $P - hn \geq 2$ . For  $1 \leq n \leq 3$ , this follows by inspecting all cases, keeping in mind that  $h < 1 \leq 2$ . Next, assume that  $n \geq 4$ . By hypothesis, and since  $n_0 \geq 2$ ,  $(2/3)\sqrt{n} \leq (2/3)(n_0 + 1) \leq n_0 < 2/h$  so that  $hn < 3\sqrt{n}$ . Hence, by Assertion 4.A,  $hn < 4\sqrt{n} - \sqrt{n} \leq P - 2$ .  $\square$

Let  $A(\sigma) = \{x \in \mathbb{Z}^2 : \sigma_x \neq -1\}$ ,  $\sigma \in \{-1, 0, 1\}^{\mathbb{Z}^2}$ . Denote by  $\mathfrak{B}$  the boundary of the valley of  $-1$  formed by the set of configurations with  $n_0(n_0 + 1)$  sites with spins different from  $-1$ :

$$\mathfrak{B} = \{\sigma \in \{-1, 0, 1\}^{\mathbb{Z}^2} : |A(\sigma)| = n_0(n_0 + 1)\}.$$

Sometimes, we consider  $\mathfrak{B}$  as a subset of  $\Omega$ . Denote by  $\mathfrak{R}$  the subset of  $\mathfrak{B}$  given by

$$\mathfrak{R} = \{\sigma \in \{-1, 0\}^{\mathbb{Z}^2} : A(\sigma) \text{ is a } n_0 \times (n_0 + 1) \text{ rectangle}\}.$$

Note that the spins of a configuration  $\sigma \in \mathfrak{R}$  are either  $-1$  or  $0$  and that all configurations in  $\mathfrak{R}$  have the same energy.

**Assertion 4.D.** *We have that  $\mathbb{H}(\sigma) \geq \mathbb{H}(\zeta) + 2$  for all  $\sigma \in \mathfrak{B} \setminus \mathfrak{R}$ ,  $\zeta \in \mathfrak{R}$ .*

*Proof.* Fix a configuration  $\sigma \in \mathfrak{B}$ . Let  $\sigma^*$  be the configuration of  $\{-1, 0\}^{\mathbb{Z}^2}$  obtained from  $\sigma$  by switching all  $+1$  spins to 0. Applying Assertion 4.C  $k$  times, where  $k$  is the number of connected components formed by  $+1$  spins, we obtain that  $\mathbb{H}(\sigma) \geq \mathbb{H}(\sigma^*) + 2k$ . It is therefore enough to prove the lemma for configurations  $\sigma \in \{-1, 0\}^{\mathbb{Z}^2}$ .

Let  $\sigma$  be a configuration in  $\mathfrak{B} \cap \{-1, 0\}^{\mathbb{Z}^2}$ . If  $A(\sigma)$  is not a connected set, by gluing the connected components of  $A(\sigma)$ , we reach a new configuration  $\sigma^* \in \mathfrak{B} \cap \{-1, 0\}^{\mathbb{Z}^2}$  such that  $A(\sigma^*) \in \mathcal{C}_m$ ,  $m = n_0(n_0 + 1)$ . Since by gluing two components, the volume remains unchanged, but the perimeter decreases at least by 2,  $\mathbb{H}(\sigma) \geq \mathbb{H}(\sigma^*) + 2$ . It is therefore enough to prove the lemma for those configuration in  $\mathfrak{B} \cap \{-1, 0\}^{\mathbb{Z}^2}$  for which  $A(\sigma)$  is a connected set.

Finally, fix a configuration in  $\mathfrak{B} \cap \{-1, 0\}^{\mathbb{Z}^2}$  for which  $A(\sigma)$  is a connected set different from a  $n_0 \times (n_0 + 1)$  rectangle. Since all spins of  $\sigma$  are either  $-1$  or 0. By definition of the energy,  $\mathbb{H}(\sigma) - \mathbb{H}(\zeta) = P(A(\sigma)) - P(A(\zeta))$ , and the result follows from Assertion 4.B.  $\square$

Denote by  $\mathfrak{R}^+$  the set of configurations in  $\{-1, 0, +1\}^{\mathbb{Z}^2}$  in which there are  $n_0(n_0 + 1) + 1$  spins which are not equal to  $-1$ . Of these spins,  $n_0(n_0 + 1)$  form a  $n_0 \times (n_0 + 1)$ -rectangle of 0 spins. The remaining spin not equal to  $-1$  is either 0 or  $+1$ .

It is clear that starting from  $-\mathbf{1}$  the set  $(\mathfrak{B} \setminus \mathfrak{R}) \cup \mathfrak{R}^+$  is hit before the chain attains the set  $\{\mathbf{0}, +\mathbf{1}\}$ :

$$H_{\mathfrak{B}^+} < H_{\{\mathbf{0}, +\mathbf{1}\}} \quad \mathbb{P}_{-\mathbf{1}} \text{ a.s.}, \quad \text{where } \mathfrak{B}^+ = (\mathfrak{B} \setminus \mathfrak{R}) \cup \mathfrak{R}^+. \quad (4.1)$$

Let  $\mathfrak{R}^a \subset \mathfrak{R}^+$  be the set of configurations for which the remaining spin is a 0 spin attached to one of the sides of the rectangle. Note that all configurations of  $\mathfrak{R}^a$  have the same energy and that  $\mathbb{H}(\xi) = \mathbb{H}(\zeta) + 2 - h$  if  $\xi \in \mathfrak{R}^a$ ,  $\zeta \in \mathfrak{R}$ . In particular,

$$\mathbb{H}(\sigma) \geq \mathbb{H}(\xi) + h, \quad \sigma \in \mathfrak{B} \setminus \mathfrak{R}, \quad \xi \in \mathfrak{R}^a. \quad (4.2)$$

On the other hand, for a configuration  $\eta \in \mathfrak{R}^+ \setminus \mathfrak{R}^a$ ,  $\mathbb{H}(\eta) \geq \mathbb{H}(\zeta) + 4 - h$  if  $\zeta \in \mathfrak{R}$ , so that

$$\mathbb{H}(\eta) \geq \mathbb{H}(\xi) + 2, \quad \eta \in \mathfrak{R}^+ \setminus \mathfrak{R}^a, \quad \xi \in \mathfrak{R}^a. \quad (4.3)$$

Recall the notation introduced just above Remark 3.2. For two disjoint subsets  $\mathcal{A}$  to  $\mathcal{B}$  of  $\Omega$ , denote by  $\xi_{\mathcal{A}, \mathcal{B}}$ , the configuration with highest energy in the optimal path joining  $\mathcal{A}$  to  $\mathcal{B}$ . By Remark 3.2 and (3.6), the limit

$$\lim_{\beta \rightarrow \infty} \frac{\text{cap}(\mathcal{A}, \mathcal{B})}{\mu_\beta(\xi_{\mathcal{A}, \mathcal{B}})} \text{ exists and takes value in } (0, \infty). \quad (4.4)$$

In particular, for every subset  $\mathcal{B}$  of  $\Omega$  and every configuration  $\sigma \notin \mathcal{B}$ ,

$$\text{cap}(\sigma, \mathcal{B}) \leq \frac{1}{Z_\beta} e^{-\beta \mathbb{H}(\sigma)}. \quad (4.5)$$

**Assertion 4.E.** For all configurations  $\xi \in \mathfrak{R}^a$ ,  $\text{cap}(\xi, -\mathbf{1}) \approx Z_\beta^{-1} \exp\{-\beta \mathbb{H}(\xi)\}$ .

*Proof.* Fix  $\xi \in \mathfrak{R}^a$ . By (4.4), (4.5), it is enough to exhibit a path  $\gamma = (\xi = \xi_0, \xi_1, \dots, \xi_n = -\mathbf{1})$  from  $\xi$  to  $-\mathbf{1}$  such that  $\max_i \mathbb{H}(\xi_i) = \mathbb{H}(\xi)$ .

Consider the path  $\gamma = (\xi = \xi_0, \xi_1, \dots, \xi_n = -\mathbf{1})$ ,  $n = n_0(n_0 + 1) + 1$ , constructed as follows.  $\xi_1$  is the configuration obtained from  $\xi$  by switching the attached particle

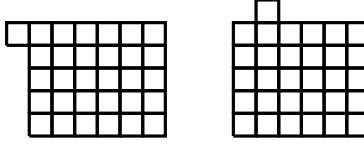


FIGURE 1. Examples of configurations  $\sigma \in \mathfrak{R}^c$  and  $\sigma' \in \mathfrak{R}^i$  in the case where  $n_0 = 5$ . A  $1 \times 1$  square centered at  $x$  has been placed at each site  $x$  occupied by a 0-spin. All the other spins are equal to  $-1$ .

from 0 to  $-1$ . Clearly,  $\mathbb{H}(\xi_1) = \mathbb{H}(\xi) - 2 + h$ , and  $\xi_1$  consists of a  $n_0 \times (n_0 + 1)$  rectangle of 0 spins.

The portion  $(\xi_1, \dots, \xi_{n_0+1})$  of the path  $\gamma$  is constructed by flipping, successively, from 0 to  $-1$ , all spins of one of the shortest sides of the rectangle, keeping until the last step the perimeter of the set  $A(\xi_i)$  equal to  $4n_0(n_0 + 1)$ . In particular,  $\xi_{n_0+1}$  consists of a  $n_0 \times n_0$  square of 0 spins,  $\mathbb{H}(\xi_{i+1}) = \mathbb{H}(\xi_i) + h$  for  $1 \leq i < n_0$ , and  $\mathbb{H}(\xi_{n_0+1}) = \mathbb{H}(\xi_{n_0}) - 2 + h$ . The energy of this piece of the path attains its maximum at  $\xi_{n_0}$  and  $\mathbb{H}(\xi_{n_0}) = \mathbb{H}(\xi_1) + (n_0 - 1)h = \mathbb{H}(\xi) - 2 + n_0h < \mathbb{H}(\xi)$ .

The path proceed in this way by always flipping from 0 to  $-1$  all spins of one of the shortest sides. It is easy to check that  $\mathbb{H}(\xi_i) < \mathbb{H}(\xi)$  for all  $1 \leq i \leq n$ , proving the assertion.  $\square$

**Assertion 4.F.** For every  $\sigma \in \mathfrak{B}^+ \setminus \mathfrak{R}^a$ ,  $\lim_{\beta \rightarrow \infty} \mathbb{P}_{-1}[H_\sigma = H_{\mathfrak{B}^+}] = 0$ .

*Proof.* Fix  $\sigma \in \mathfrak{B}^+ \setminus \mathfrak{R}^a$  and  $\xi \in \mathfrak{R}^a$ . By (3.3); by the monotonicity of the capacity, stated in [17, Lemma 2.2], and by (4.5); and by Assertion 4.E,

$$\mathbb{P}_{-1}[H_\sigma = H_{\mathfrak{B}^+}] \leq \frac{\text{cap}(\sigma, -1)}{\text{cap}(\mathfrak{B}^+, -1)} \leq \frac{C_0}{Z_\beta} \frac{e^{-\beta \mathbb{H}(\sigma)}}{\text{cap}(\xi, -1)} \leq C_0 e^{-\beta \{\mathbb{H}(\sigma) - \mathbb{H}(\xi)\}}$$

for some finite constant  $C_0$  independent of  $\beta$ . By (4.2), (4.3), this expression vanishes as  $\beta \uparrow \infty$ , proving the assertion.  $\square$

Denote by  $\mathcal{S}$  the set of stable configurations:

$$\mathcal{S} = \left\{ \sigma \in \Omega : \lim_{\beta \rightarrow \infty} \lambda_\beta(\sigma) = 0 \right\}. \quad (4.6)$$

The next assertion is the only one in which capacities are not used to derive the needed bounds, because we estimate the probability of reaching a state which can be attained through paths in which the energy never increase. The argument, though, is fairly simple.

Denote by  $\mathfrak{R}^c, \mathfrak{R}^i$  the configurations of  $\mathfrak{R}^a$  in which the extra particle is attached to the corner, interior of the rectangle, respectively (cf. Figure 1).

**Assertion 4.G.** For  $\sigma \in \mathfrak{R}^c$  and  $\sigma' \in \mathfrak{R}^i$ ,

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma[H_{\sigma_+} = H_{\mathcal{S}}] &= 1/2 \quad \text{and} \quad \lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma[H_{\sigma_-} = H_{\mathcal{S}}] = 1/2, \\ \lim_{\beta \rightarrow \infty} \mathbb{P}_{\sigma'}[H_{\sigma_+} = H_{\mathcal{S}}] &= 2/3 \quad \text{and} \quad \lim_{\beta \rightarrow \infty} \mathbb{P}_{\sigma'}[H_{\sigma_-} = H_{\mathcal{S}}] = 1/3, \end{aligned}$$

where  $\sigma_-$  is the configuration obtained from  $\sigma$  or  $\sigma'$  by flipping to  $-1$  the attached 0 spin, and  $\sigma_+$  is the configuration whose set  $A(\sigma_+)$ , formed only by 0 spins, is the smallest rectangle which contains  $A(\sigma)$ .

*Proof.* Suppose that the extra 0 spin is not attached to the corner of the rectangle. Denote by  $\sigma_1, \sigma_2$  the configurations obtained from  $\sigma'$  by flipping from  $-1$  to  $0$  one of the two  $-1$  spins which has two neighbor spins equal to  $0$ , and let  $\sigma_0 = \sigma_-$ . By definition,  $R_\beta(\sigma', \sigma_j) = 1$ ,  $0 \leq j \leq 2$ , and  $R_\beta(\sigma', \sigma') = o(1)$  for all the other configurations, where  $o(1)$  represents an expression which vanishes as  $\beta \uparrow \infty$ . This shows that  $p_\beta(\sigma', \sigma_j)$  converges to  $1/3$ ,  $0 \leq j \leq 2$ . We may repeat this argument to show that from  $\sigma_j$ ,  $j = 1, 2$ , one reaches  $\mathcal{S}$  at  $\sigma_+$  with a probability asymptotically equal to  $1$ . The argument is similar if the extra spin is attached to the corner.  $\square$

**Assertion 4.H.** Fix a configuration  $\sigma \in \mathcal{S}$  for which  $A(\sigma)$  is a  $m \times n$  rectangle of  $0$  spins in a sea of  $-1$  spins. Assume that  $m \leq n$ . Then,

$$\lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma[H_{\mathcal{B}} = H_{\mathcal{S} \setminus \{\sigma\}}] = 1.$$

In this equation, if  $n_0 < m$ ,  $n \leq L-3$ ,  $\mathcal{B}$  is the set of four configurations in which a row or a column of  $0$  spins is added to the rectangle  $A(\sigma)$ . If  $n_0 < m < n = L-2$ , the set  $\mathcal{B}$  is a triple which includes a band of  $0$  spins of width  $m$  and two configurations in which a row or a column of  $0$  spins of length  $n$  is added to the rectangle  $A(\sigma)$ . If  $n_0 < m \leq L-3$ ,  $n = L$ , the set  $\mathcal{B}$  is a pair formed by two bands of  $0$  spins of width  $m+1$ . If  $n_0 < m = n = L-2$ ,  $\mathcal{B}$  is a pair of two bands of width  $L-2$ . If  $n_0 < m = L-2$ ,  $n = L$ ,  $\mathcal{B} = \{\mathbf{0}\}$ . Finally, if  $2 \leq m \leq n_0$ ,  $n \geq 3$ , the set  $\mathcal{B}$  is the pair (quaternion if  $m = n$ ) of configurations in which a row or a column of  $0$  spins of length  $m$  is removed from the rectangle  $A(\sigma)$ , and if  $m = n = 2$ ,  $\mathcal{B} = \{-1\}$ .

*Proof.* The assertion follows from inequality (3.3), and from estimates of  $\text{cap}(\sigma, \mathcal{S} \setminus \{\sigma\})$ ,  $\text{cap}(\sigma, \mathcal{S} \setminus [\mathcal{B} \cup \{\sigma\}])$ . In the case  $m > n_0$ ,  $\text{cap}(\sigma, \mathcal{S} \setminus \{\sigma\}) \approx \mu_\beta(\sigma) \exp\{-\beta(2-h)\}$  and  $\text{cap}(\sigma, \mathcal{S} \setminus [\mathcal{B} \cup \{\sigma\}]) \prec \mu_\beta(\sigma) \exp\{-\beta(2-h)\}$ , while in the case  $m \leq n_0$ ,  $\text{cap}(\sigma, \mathcal{S} \setminus \{\sigma\}) \approx \mu_\beta(\sigma) \exp\{-\beta(m-1)h\}$  and  $\text{cap}(\sigma, \mathcal{S} \setminus [\mathcal{B} \cup \{\sigma\}]) \prec \mu_\beta(\sigma) \exp\{-\beta(m-1)h\}$ .  $\square$

**Lemma 4.1.** For every  $\sigma \in \mathfrak{B}^+$ ,

$$\lim_{\beta \rightarrow \infty} \mathbb{P}_{-1}[H_\sigma = H_{\mathfrak{B}^+}] = \frac{1}{|\mathfrak{R}^a|} \mathbf{1}\{\sigma \in \mathfrak{R}^a\},$$

where  $|\mathfrak{R}^a|$  represents the number of configurations in  $\mathfrak{R}^a$ .

*Proof.* In view of Assertion 4.F, we may restrict our attention to  $\sigma \in \mathfrak{R}^a$ . Fix a reference configuration  $\sigma^*$  in  $\mathfrak{R}^a$ . By (3.1) and by definition of the capacity,

$$\mathbb{P}_{-1}[H_\sigma = H_{\mathfrak{B}^+}] = \frac{M(-1) \mathbb{P}_{-1}[H_\sigma = H_{\mathfrak{B}^+ \cup \{-1\}}^+]}{\text{cap}(-1, \mathfrak{B}^+)}.$$

By reversibility, the numerator of this expression is equal to

$$M(\sigma) \mathbb{P}_\sigma[H_{-1} = H_{\mathfrak{B}^+ \cup \{-1\}}^+] = \mu_\beta(\sigma) \lambda(\sigma) \mathbb{P}_\sigma[H_{-1} = H_{\mathfrak{B}^+ \cup \{-1\}}^+].$$

By Assertions 4.G and 4.H,  $\mathbb{P}_\sigma[H_{-1} = H_{\mathfrak{B}^+ \cup \{-1\}}^+] = \mathbf{n}(\sigma) + o(1)$ , where

$$\mathbf{n}(\sigma) = \begin{cases} 1/2 & \text{if } \sigma \in \mathfrak{R}^c, \\ 1/3 & \text{if } \sigma \in \mathfrak{R}^i. \end{cases} \quad \text{Since } \lambda(\sigma) = \begin{cases} 2 + o(1) & \text{if } \sigma \in \mathfrak{R}^c, \\ 3 + o(1) & \text{if } \sigma \in \mathfrak{R}^i. \end{cases}$$

Since  $\mu_\beta(\sigma) = \mu_\beta(\sigma^*)$ , we conclude that

$$\mathbb{P}_{-1}[H_\sigma = H_{\mathfrak{B}^+}] = \frac{\mu_\beta(\sigma^*)}{\text{cap}(-1, \mathfrak{B}^+)} \left( \mathbf{1}\{\sigma \in \mathfrak{R}^a\} + o(1) \right).$$

Summing over  $\sigma \in \mathfrak{B}^+$ , we conclude that  $\mu_\beta(\sigma^*)/\text{cap}(-\mathbf{1}, \mathfrak{B}^+) = |\mathfrak{R}^a|^{-1}(1 + o(1))$ , which completes the proof of the assertion.  $\square$

It follows from the proof of the previous lemma that for any configuration  $\sigma^* \in \mathfrak{R}^a$ ,

$$\lim_{\beta \rightarrow \infty} \frac{\text{cap}(-\mathbf{1}, \mathfrak{B}^+)}{\mu_\beta(\sigma^*)} = |\mathfrak{R}^a|. \quad (4.7)$$

Denote by  $\mathfrak{R}^l, \mathfrak{R}^s$  the configurations of  $\mathfrak{R}^a$  in which the extra particle is attached to one of the longest, shortest sides, respectively, and let  $\mathfrak{R}^{lc} = \mathfrak{R}^l \cap \mathfrak{R}^c$ ,  $\mathfrak{R}^{li} = \mathfrak{R}^l \cap \mathfrak{R}^i$ . The next lemma is an immediate consequence of Assertions 4.G and 4.H.

**Lemma 4.2.** *For  $\sigma \in \mathfrak{R}^{lc}$ ,  $\sigma' \in \mathfrak{R}^{li}$ , and  $\sigma'' \in \mathfrak{R}^s$ ,*

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma[H_{-1} = H_{\mathcal{M}}] &= 1/2 \quad \text{and} \quad \lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma[H_{\mathbf{0}} = H_{\mathcal{M}}] = 1/2, \\ \lim_{\beta \rightarrow \infty} \mathbb{P}_{\sigma'}[H_{-1} = H_{\mathcal{M}}] &= 1/3 \quad \text{and} \quad \lim_{\beta \rightarrow \infty} \mathbb{P}_{\sigma'}[H_{\mathbf{0}} = H_{\mathcal{M}}] = 2/3, \\ \lim_{\beta \rightarrow \infty} \mathbb{P}_{\sigma''}[H_{-1} = H_{\mathcal{M}}] &= 1. \end{aligned}$$

It follows from this assertion that for every  $\sigma \in \mathfrak{R}^a$ ,

$$\lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma[H_{\{-1, \mathbf{0}\}} < H_{+1}] = 1, \quad (4.8)$$

*Proof of Proposition 2.3, Part A.* We first claim that

$$\text{cap}(-\mathbf{1}, \{\mathbf{0}, +1\}) = \text{cap}(-\mathbf{1}, \mathfrak{B}^+) \sum_{\sigma \in \mathfrak{B}^+} \mathbb{P}_{-1}[H_\sigma = H_{\mathfrak{B}^+}] \mathbb{P}_\sigma[H_{\{\mathbf{0}, +1\}} < H_{-1}]. \quad (4.9)$$

Indeed, since starting from  $-\mathbf{1}$  the process hits  $\mathfrak{B}^+$  before  $\{\mathbf{0}, +1\}$ , by the strong Markov property we have that

$$\mathbb{P}_{-1}[H_{\{\mathbf{0}, +1\}} < H_{-1}^+] = \sum_{\sigma \in \mathfrak{B}^+} \mathbb{P}_{-1}[H_\sigma = H_{\mathfrak{B}^+ \cup \{-1\}}^+] \mathbb{P}_\sigma[H_{\{\mathbf{0}, +1\}} < H_{-1}].$$

By (3.1), we may rewrite the previous expression as

$$\mathbb{P}_{-1}[H_{\mathfrak{B}^+} < H_{-1}^+] \sum_{\sigma \in \mathfrak{B}^+} \mathbb{P}_{-1}[H_\sigma = H_{\mathfrak{B}^+}] \mathbb{P}_\sigma[H_{\{\mathbf{0}, +1\}} < H_{-1}].$$

This proves (4.9) in view of the definition (2.5) of the capacity.

By (4.9) and (4.7), for any configuration  $\sigma^* \in \mathfrak{R}^a$ ,

$$\lim_{\beta \rightarrow \infty} \frac{\text{cap}(-\mathbf{1}, \{\mathbf{0}, +1\})}{\mu_\beta(\sigma^*)} = |\mathfrak{R}^a| \lim_{\beta \rightarrow \infty} \sum_{\sigma \in \mathfrak{B}^+} \mathbb{P}_{-1}[H_\sigma = H_{\mathfrak{B}^+}] \mathbb{P}_\sigma[H_{\{\mathbf{0}, +1\}} < H_{-1}].$$

By Lemma 4.1, the right hand side is equal to

$$\lim_{\beta \rightarrow \infty} \sum_{\sigma \in \mathfrak{R}^a} \mathbb{P}_\sigma[H_{\{\mathbf{0}, +1\}} < H_{-1}]. \quad (4.10)$$

By Lemma 4.2, this expression is equal to  $(1/2)|\mathfrak{R}^{lc}| + (2/3)|\mathfrak{R}^{li}| = 2|\Lambda_L|\{2 + (4/3)(n_0 - 1)\}$ , which completes the proof of the first claim of the proposition.  $\square$

**Assertion 4.I.** *We have that*

$$\lim_{\beta \rightarrow \infty} \frac{\text{cap}(-\mathbf{1}, \mathbf{0})}{\text{cap}(-\mathbf{1}, \{\mathbf{0}, +1\})} = 1.$$

*Proof.* Let  $\sigma^*$  be a configuration in  $\mathfrak{R}^a$ . By the proof of Proposition 2.3 up to (4.10),

$$\lim_{\beta \rightarrow \infty} \frac{\text{cap}(-\mathbf{1}, \mathbf{0})}{\mu_\beta(\sigma^*)} = \lim_{\beta \rightarrow \infty} \sum_{\sigma \in \mathfrak{R}^a} \mathbb{P}_\sigma[H_{\mathbf{0}} < H_{-\mathbf{1}}] .$$

By (4.8), this expression is equal to (4.10). This completes the proof of the assertion.  $\square$

*Proof of Proposition 2.1.* Let  $q(\sigma) = \mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}]$ ,  $\sigma \in \mathfrak{B}^+$ . By Assertion 4.F,  $q(\sigma) \rightarrow 0$  if  $\sigma \notin \mathfrak{R}^a$ . Hence, by (4.1),

$$\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] = \sum_{\sigma \in \mathfrak{B}^+} q(\sigma) \mathbb{P}_\sigma[H_{+\mathbf{1}} < H_{\mathbf{0}}] = \sum_{\sigma \in \mathfrak{R}^a} q(\sigma) \mathbb{P}_\sigma[H_{+\mathbf{1}} < H_{\mathbf{0}}] + o(1) ,$$

By (4.8), for all  $\sigma \in \mathfrak{R}^a$ ,

$$\mathbb{P}_\sigma[H_{+\mathbf{1}} < H_{\mathbf{0}}] = \mathbb{P}_\sigma[H_{+\mathbf{1}} < H_{\mathbf{0}}, H_{\{-1, \mathbf{0}\}} < H_{+\mathbf{1}}] = \mathbb{P}_\sigma[H_{-\mathbf{1}} < H_{+\mathbf{1}} < H_{\mathbf{0}}] .$$

Therefore, by the strong Markov property,

$$\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] = \mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] \sum_{\sigma \in \mathfrak{R}^a} q(\sigma) \mathbb{P}_\sigma[H_{-\mathbf{1}} < H_{\{\mathbf{0}, +\mathbf{1}\}}] + o(1) .$$

By Lemma 4.2, for  $\sigma \in \mathfrak{R}^l$ ,  $\limsup_{\beta \rightarrow \infty} \mathbb{P}_\sigma[H_{-\mathbf{1}} < H_{\{\mathbf{0}, +\mathbf{1}\}}] \leq 1/2$ , which completes the proof of the proposition.  $\square$

*Proof of Proposition 2.2.* We prove the proposition when the chain starts from  $-\mathbf{1}$ , the argument being analogous when it starts from  $\mathbf{0}$ . Since the chain hits  $\mathfrak{B}^+$  before reaching  $\mathbf{0}$  and  $\mathfrak{R}^l$ , by the strong Markov property,

$$\mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{R}^l} < H_{\mathbf{0}}] = \sum_{\sigma \in \mathfrak{B}^+} \mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}] \mathbb{P}_\sigma[H_{\mathfrak{R}^l} < H_{\mathbf{0}}] .$$

By Lemma 4.1, this expression is equal to

$$(1 + o(1)) \frac{1}{|\mathfrak{R}^a|} \left\{ |\mathfrak{R}^l| + \sum_{\sigma \in \mathfrak{R}^s} \mathbb{P}_\sigma[H_{\mathfrak{R}^l} < H_{\mathbf{0}}] \right\} .$$

By Assertions 4.G and 4.H, for all  $\sigma \in \mathfrak{R}^s$ ,  $\sigma' \in \mathfrak{R}$

$$\lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma[H_{\mathfrak{R}} < H_{\mathfrak{R}^l \cup \{-1, \mathbf{0}\}}] = 1 , \quad \lim_{\beta \rightarrow \infty} \mathbb{P}_{\sigma'}[H_{-\mathbf{1}} < H_{\mathfrak{R}^l \cup \{\mathbf{0}\}}] = 1 .$$

Therefore, for all  $\sigma \in \mathfrak{R}^s$ ,

$$\lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma[H_{-\mathbf{1}} < H_{\mathfrak{R}^l \cup \{\mathbf{0}\}}] = 1 .$$

Hence, by the strong Markov property and by the first two identities of this proof,

$$\mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{R}^l} < H_{\mathbf{0}}] = (1 + o(1)) \frac{1}{|\mathfrak{R}^a|} \left\{ |\mathfrak{R}^l| + \sum_{\sigma \in \mathfrak{R}^s} \mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{R}^l} < H_{\mathbf{0}}] \right\} ,$$

which completes the proof of the proposition.  $\square$

## 5. PROOFS OF PROPOSITIONS 2.3.B AND 2.4

We examine in this section the metastable behavior of the Blume-Capel model starting from  $\mathbf{0}$ . The main observation is that the energy barrier from  $\mathbf{0}$  to  $-\mathbf{1}$  is larger than the one from  $\mathbf{0}$  to  $+\mathbf{1}$ . We may therefore ignore  $-\mathbf{1}$  and argue by symmetry that the passage from  $\mathbf{0}$  to  $+\mathbf{1}$  is identical to the one from  $-\mathbf{1}$  to  $\mathbf{0}$ .

In analogy to the notation introduced right before Assertion 4.D, let  $\mathfrak{B}_0$  be the set of configurations with  $n_0(n_0 + 1)$  sites with spins different from 0, and let  $\mathfrak{R}_0$  be the subset of  $\mathfrak{B}_0$  given by

$$\mathfrak{R}_0 = \{ \sigma \in \{0, +1\}^{\mathbb{Z}^2} : \{x : \sigma(x) \neq 0\} \text{ forms a } n_0 \times (n_0 + 1) \text{ rectangle} \} .$$

Denote by  $\mathfrak{R}_0^+$  the set of configurations in  $\{-1, 0, +1\}^{\mathbb{Z}^2}$  in which there are  $n_0(n_0 + 1) + 1$  spins which are not equal to 0, and in which  $n_0(n_0 + 1)$  spins of magnetization  $+1$  form a  $n_0 \times (n_0 + 1)$ -rectangle. Finally, let  $\mathfrak{R}_0^a \subset \mathfrak{R}_0^+$  be the set of configurations for which the remaining spin is a  $+1$  spin attached to one of the sides of the rectangle, and let

$$\mathfrak{B}_0^+ = (\mathfrak{B}_0 \setminus \mathfrak{R}_0) \cup \mathfrak{R}_0^+ .$$

**Assertion 5.A.** *For every  $\sigma \in \mathfrak{B}_0^+$ ,  $\lim_{\beta \rightarrow \infty} \mathbb{P}_0[H_\sigma = H_{\mathfrak{B}_0^+}] = |\mathfrak{R}_0^a|^{-1} \mathbf{1}\{\sigma \in \mathfrak{R}_0^a\}$ . Moreover, if  $\sigma^*$  represents a configuration in  $\mathfrak{R}_0^a$ ,*

$$\lim_{\beta \rightarrow \infty} \frac{\text{cap}(\mathbf{0}, \mathfrak{B}_0^+)}{\mu_\beta(\sigma^*)} = |\mathfrak{R}_0^a| .$$

*Proof.* As in Assertion 4.F, we may exclude all configurations  $\sigma \in \{0, +1\}^{\Lambda_L}$  which do not belong to  $\mathfrak{R}_0^a$ . We may also exclude all configurations in  $\mathfrak{B}_0^+$  which have a negative spin since by turning all negative spins into positive spins we obtain a new configurations whose energy is strictly smaller than the one of the original configuration. For the configurations in  $\mathfrak{R}_0^a$  we may apply the arguments presented in the proof of Lemma 4.1.  $\square$

Denote by  $\mathfrak{R}_0^c$ ,  $\mathfrak{R}_0^i$  the configurations of  $\mathfrak{R}_0^a$  in which the extra particle is attached to the corner, interior of the rectangle, respectively. Denote by  $\mathfrak{R}_0^l$ ,  $\mathfrak{R}_0^s$  the configurations of  $\mathfrak{R}_0^a$  in which the extra particle is attached to one of the longest, shortest sides, respectively, and let  $\mathfrak{R}_0^{lc} = \mathfrak{R}_0^l \cap \mathfrak{R}_0^c$ ,  $\mathfrak{R}_0^{li} = \mathfrak{R}_0^l \cap \mathfrak{R}_0^i$ . The proof of the next assertion is analogous to the one of Lemma 4.2 since it concerns configurations with only 0 and  $+1$  spins.

**Assertion 5.B.** *For  $\sigma \in \mathfrak{R}_0^{lc}$ ,  $\sigma' \in \mathfrak{R}_0^{li}$ , and  $\sigma'' \in \mathfrak{R}_0^s$ ,*

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma[H_{\mathbf{0}} = H_{\mathcal{M}}] &= 1/2 \quad \text{and} \quad \lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma[H_{+\mathbf{1}} = H_{\mathcal{M}}] = 1/2 , \\ \lim_{\beta \rightarrow \infty} \mathbb{P}_{\sigma'}[H_{\mathbf{0}} = H_{\mathcal{M}}] &= 1/3 \quad \text{and} \quad \lim_{\beta \rightarrow \infty} \mathbb{P}_{\sigma'}[H_{+\mathbf{1}} = H_{\mathcal{M}}] = 2/3 , \\ \lim_{\beta \rightarrow \infty} \mathbb{P}_{\sigma''}[H_{\mathbf{0}} = H_{\mathcal{M}}] &= 1 . \end{aligned}$$

The next claim follows from the previous two assertions.

**Assertion 5.C.** *We have that*

$$\lim_{\beta \rightarrow \infty} \mathbb{P}_0[H_{-\mathbf{1}} < H_{+\mathbf{1}}] = 0 .$$



*Proof.* Since, starting from  $\mathbf{0}$ , the set  $\mathfrak{B}_0^+$  is reached before the process hits  $\{-1, +1\}$ , by the strong Markov property,

$$\lim_{\beta \rightarrow \infty} \mathbb{P}_0[H_{-1} < H_{+1}] = \lim_{\beta \rightarrow \infty} \sum_{\sigma \in \mathfrak{B}_0^+} \mathbb{P}_0[H_\sigma = H_{\mathfrak{B}_0^+}] \mathbb{P}_\sigma[H_{-1} < H_{+1}].$$

By Assertion 5.A and by the strong Markov property at time  $H_{\mathcal{M}}$ , this expression is equal to

$$\lim_{\beta \rightarrow \infty} \frac{1}{|\mathfrak{R}_0^a|} \sum_{\sigma \in \mathfrak{R}_0^a} \mathbb{E}_\sigma[\mathbb{P}_{\sigma(H_{\mathcal{M}})}[H_{-1} < H_{+1}]] = c_0 \lim_{\beta \rightarrow \infty} \mathbb{P}_0[H_{-1} < H_{+1}].$$

where we applied Assertion 5.B to derive the last identity. In this equation,  $c_0 = \{|\mathfrak{R}_0^{lc}|/2|\mathfrak{R}_0^a|\} + \{|\mathfrak{R}_0^{li}|/3|\mathfrak{R}_0^a|\} < 1$ . This completes the proof of the assertion.  $\square$

*Proof of Proposition 2.3, Part B.* The proof is similar to the one of Part A, presented in the previous section. As in (4.9), we have that

$$\text{cap}(\mathbf{0}, \{-1, +1\}) = \text{cap}(\mathbf{0}, \mathfrak{B}_0^+) \sum_{\sigma \in \mathfrak{B}_0^+} \mathbb{P}_0[H_\sigma = H_{\mathfrak{B}_0^+}] \mathbb{P}_\sigma[H_{\{-1, +1\}} < H_0].$$

Hence, by Assertion 5.A, for any configuration  $\sigma^* \in \mathfrak{R}_0^a$ ,

$$\lim_{\beta \rightarrow \infty} \frac{\text{cap}(\mathbf{0}, \{-1, +1\})}{\mu_\beta(\sigma^*)} = |\mathfrak{R}_0^a| \lim_{\beta \rightarrow \infty} \sum_{\sigma \in \mathfrak{B}_0^+} \mathbb{P}_0[H_\sigma = H_{\mathfrak{B}_0^+}] \mathbb{P}_\sigma[H_{\{-1, +1\}} < H_0].$$

By Assertion 5.A, the right hand side is equal to

$$\lim_{\beta \rightarrow \infty} \sum_{\sigma \in \mathfrak{R}_0^a} \mathbb{P}_\sigma[H_{\{-1, +1\}} < H_0].$$

By Assertion 5.B, this expression is equal to  $(1/2)|\mathfrak{R}_0^{lc}| + (2/3)|\mathfrak{R}_0^{li}| = 2|\Lambda_L|\{2 + (4/3)(n_0 - 1)\}$ , which completes the proof of the second claim of the proposition.  $\square$

As in Assertion 4.I we have that

$$\lim_{\beta \rightarrow \infty} \frac{\text{cap}(\mathbf{0}, +1)}{\text{cap}(\mathbf{0}, \{-1, +1\})} = 1. \quad (5.1)$$

**Assertion 5.D.** *We have that*

$$\lim_{\beta \rightarrow \infty} \frac{\text{cap}(-1, +1)}{\text{cap}(-1, \{\mathbf{0}, +1\})} = 1.$$

*Proof.* We repeat the proof of the part A of Proposition 2.3 up (4.10) to obtain that

$$\lim_{\beta \rightarrow \infty} \frac{\text{cap}(-1, +1)}{\mu_\beta(\sigma^*)} = \lim_{\beta \rightarrow \infty} \sum_{\sigma \in \mathfrak{R}^a} \mathbb{P}_\sigma[H_{+1} < H_{-1}],$$

if  $\sigma^*$  represents a configuration in  $\mathfrak{R}^a$ . By (4.8), this expression is equal to

$$\begin{aligned} & \lim_{\beta \rightarrow \infty} \sum_{\sigma \in \mathfrak{R}^a} \mathbb{P}_\sigma[H_0 < H_{+1} < H_{-1}] \\ &= \lim_{\beta \rightarrow \infty} \mathbb{P}_0[H_{+1} < H_{-1}] \sum_{\sigma \in \mathfrak{R}^a} \mathbb{P}_\sigma[H_0 < H_{\{-1, +1\}}]. \end{aligned}$$

where we used the strong Markov property in the last step. By Lemma 4.2 and Assertion 5.C, this limit is equal to  $(1/2)|\mathfrak{R}^{lc}| + (2/3)|\mathfrak{R}^{li}|$ , which completes the proof of the assertion.  $\square$

**Assertion 5.E.** *We have that*

$$\lim_{\beta \rightarrow \infty} \frac{\text{cap}(+1, \{-1, 0\})}{\text{cap}(0, \{-1, +1\})} = 1.$$

*Proof.* Indeed, by monotonicity of the capacity and by (3.5),

$$\text{cap}(+1, 0) \leq \text{cap}(+1, \{-1, 0\}) \leq \text{cap}(+1, 0) + \text{cap}(+1, -1).$$

By Assertion 5.D, by (5.1), and by Proposition 2.3,  $\text{cap}(+1, -1)/\text{cap}(0, +1) \rightarrow 0$  as  $\beta \uparrow \infty$ . Hence,

$$\lim_{\beta \rightarrow \infty} \frac{\text{cap}(+1, \{-1, 0\})}{\text{cap}(0, +1)} = 1.$$

To complete the proof, it remains to recall (5.1).  $\square$

We turn to the proof of Proposition 2.4. We first show that the assumption of Lemma 3.3 are in force. Recall Remark 3.4.

**Assertion 5.F.** *Consider two configurations  $\sigma \notin \mathcal{M}$  and  $\eta \in \mathcal{M}$ . If  $\mathbb{H}(\sigma) \leq \mathbb{H}(\eta)$ , then  $\mathbb{H}(\xi_{\sigma, \mathcal{M}}) - \mathbb{H}(\sigma) < \mathbb{H}(\xi_{\sigma, \eta}) - \mathbb{H}(\eta)$ .*

*Proof.* We claim that for any configuration  $\sigma \notin \mathcal{M}$ ,  $\mathbb{H}(\xi_{\sigma, \mathcal{M}}) - \mathbb{H}(\sigma) \leq 2 - h$ . To prove this claim it is enough to exhibit a self-avoiding path from  $\sigma$  to  $\mathcal{M}$  whose energy is kept below  $\mathbb{H}(\sigma) + 2 - h$ . This is easy. Starting from  $\sigma$  we may first reach the set  $\mathcal{S}$  of stable configurations through a path whose energy does not increase. Denote by  $\sigma^*$  the configuration in  $\mathcal{S}$  attained through this path. From  $\sigma^*$  we may reach the set  $\mathcal{M}$  by removing all small droplets (the ones whose smaller side has length  $n_0$  or less) and by increasing the large droplets (the ones whose both sides have length at least  $n_0 + 1$ ) in such a way that the energy remains less than or equal to  $\mathbb{H}(\sigma^*) + 2 - h$ . This proves the claim.

On the other hand, since  $\mathbb{H}(\zeta) \geq \mathbb{H}(\eta) + 4 - h$  for any configuration  $\zeta$  which differs from  $\eta$  at one site,  $\mathbb{H}(\xi_{\sigma, \eta}) - \mathbb{H}(\eta) \geq 4 - h$ , which proves the assertion.  $\square$

**Assertion 5.G.** *We have that*

$$\lim_{\beta \rightarrow \infty} \frac{M(-1) \mathbb{P}_{-1}[H_0 < H_{\{-1, +1\}}^+]}{\text{cap}(-1, \{0, +1\})} = 1.$$

*Proof.* Fix  $\sigma^* \in \mathfrak{R}^a$ . In view of Proposition 2.3, it is enough to show that

$$\lim_{\beta \rightarrow \infty} \frac{M(-1) \mathbb{P}_{-1}[H_0 < H_{\{-1, +1\}}^+]}{\mu_\beta(\sigma^*)} = \frac{4(2n_0 + 1)}{3} |\Lambda_L|.$$

In the proof of Proposition 2.3.A, replace  $\text{cap}(-1, \{0, +1\})$  by the numerator appearing in the statement of this assertion. The proof is identical up to formula (4.10). It remains to estimate

$$\lim_{\beta \rightarrow \infty} \sum_{\sigma \in \mathfrak{R}^a} \mathbb{P}_\sigma[H_0 < H_{\{-1, +1\}}] \text{ which is equal to } \lim_{\beta \rightarrow \infty} \sum_{\sigma \in \mathfrak{R}^a} \mathbb{P}_\sigma[H_{\{0, +1\}} < H_{-1}]$$

in view of (4.8). This expression has been computed at the end of the proof of Proposition 2.3.A, which completes the proof of the assertion.  $\square$

*Proof of Proposition 2.4.* We first assume that the chain starts from  $0$ . By Lemma 3.3 and Assertion 5.F,

$$\mathbb{E}_0[H_{+1}] = (1 + o(1)) \frac{1}{\text{cap}(0, +1)} \left\{ \mu_\beta(0) + \mu_\beta(-1) \mathbb{P}_{-1}[H_0 < H_{+1}] \right\}.$$

Since the second term in the expression inside braces is bounded by  $\mu_\beta(-1) \prec \mu_\beta(\mathbf{0})$ , the expectation is equal to  $(1 + o(1))\mu_\beta(\mathbf{0})/\text{cap}(\mathbf{0}, +1)$ . To complete the proof it remains to recall (5.1) and Proposition 2.3.

We turn to the case in which the chain starts from  $-1$ . By Lemma 3.3,

$$\mathbb{E}_{-1}[H_{+1}] = (1 + o(1)) \frac{1}{\text{cap}(-1, +1)} \left\{ \mu_\beta(-1) + \mu_\beta(\mathbf{0}) \mathbb{P}_0[H_{-1} < H_{+1}] \right\}.$$

By (3.1), by reversibility and by definition of the capacity,

$$\begin{aligned} \mu_\beta(\mathbf{0}) \mathbb{P}_0[H_{-1} < H_{+1}] &= \frac{M(\mathbf{0}) \mathbb{P}_0[H_{-1} < H_{\{\mathbf{0}, +1\}}^+]}{\lambda_\beta(\mathbf{0}) \mathbb{P}_0[H_{\{-1, +1\}} < H_0^+]} \\ &= \frac{M(-1) \mathbb{P}_{-1}[H_0 < H_{\{-1, +1\}}^+]}{\lambda_\beta(\mathbf{0}) \mathbb{P}_0[H_{\{-1, +1\}} < H_0^+]} = \frac{\mu_\beta(\mathbf{0}) M(-1) \mathbb{P}_{-1}[H_0 < H_{\{-1, +1\}}^+]}{\text{cap}(\mathbf{0}, \{-1, +1\})}. \end{aligned}$$

Hence, by Assertion 5.G,

$$\mathbb{E}_{-1}[H_{+1}] = (1 + o(1)) \frac{\mu_\beta(-1)}{\text{cap}(-1, +1)} \left\{ 1 + \frac{\mu_\beta(\mathbf{0}) \text{cap}(-1, \{\mathbf{0}, +1\})}{\mu_\beta(-1) \text{cap}(\mathbf{0}, \{-1, +1\})} \right\}.$$

To complete the proof it remains to recall the statements of Proposition 2.3 and Assertion 5.D.  $\square$

## 6. THE HITTING TIME OF $\mathbf{0}$ STARTING FROM $-1$

We prove in this section the identities (2.7) and (2.8). We start with (2.7). By Lemma 3.3 and Assertion 5.F,

$$\mathbb{E}_{-1}[H_0] = (1 + o(1)) \frac{1}{\text{cap}(-1, \mathbf{0})} \left\{ \mu_\beta(-1) + \mu_\beta(+1) \mathbb{P}_{+1}[H_{-1} < H_0] \right\}.$$

By (3.1) and the first identity in (3.2), and by reversibility, the second term inside braces is equal to

$$\mu_\beta(+1) M(+1) \frac{\mathbb{P}_{+1}[H_{-1} < H_{\{\mathbf{0}, +1\}}^+]}{\text{cap}(+1, \{-1, \mathbf{0}\})} = \mu_\beta(+1) M(-1) \frac{\mathbb{P}_{-1}[H_{+1} < H_{\{-1, \mathbf{0}\}}^+]}{\text{cap}(+1, \{-1, \mathbf{0}\})}.$$

By the first identity in (3.2), this expression is equal to

$$\mu_\beta(+1) \frac{\text{cap}(-1, \{\mathbf{0}, +1\})}{\text{cap}(+1, \{\mathbf{0}, -1\})} \mathbb{P}_{-1}[H_{+1} < H_0].$$

By Assertion 5.E and Proposition 2.3, we may replace the ratio of the capacities by  $\mu_\beta(-1)/\mu_\beta(\mathbf{0})$ . Hence,

$$\mathbb{E}_{-1}[H_0] = (1 + o(1)) \frac{\mu_\beta(-1)}{\text{cap}(-1, \mathbf{0})} \left\{ 1 + \frac{\mu_\beta(+1)}{\mu_\beta(\mathbf{0})} \mathbb{P}_{-1}[H_{+1} < H_0] \right\}.$$

To complete the proof of (2.7), it remains to recall the definition of  $\theta_\beta$ , the statement of Assertion 4.I, and the one of Lemma 6.1 below.

**Lemma 6.1.** *We have that*

$$\lim_{\beta \rightarrow \infty} \frac{\mu_\beta(+1)}{\mu_\beta(\mathbf{0})} \mathbb{P}_{-1}[H_{+1} < H_0] = \infty.$$

The proof of this lemma is divided in several assertions. By (3.1), and by the definition of the capacity,

$$\mathbb{P}_{-1}[H_{+1} < H_0] = \frac{\mu_\beta(-1) \lambda_\beta(-1) \mathbb{P}_{-1}[H_{+1} < H_{\{-1,0\}}^+]}{\text{cap}(-1, \{-1, 0\})}. \quad (6.1)$$

We estimate the probability appearing in the numerator. This is done by proposing a path from  $-1$  to  $+1$  which does not visit  $0$ . The obvious path is the optimal one from  $-1$  to  $0$  juxtaposed with the optimal one from  $0$  to  $+1$ , modified not to visit  $0$ .

We describe the path in  $\mathcal{S}$ , the set of stable configurations introduced in (4.6). Let  $\xi_0 \in \mathcal{S}$  be the configuration formed by a  $L \times (L-2)$  band of 0-spins and a  $L \times 2$  band of  $-1$  spins. The first piece of the path, denoted by  $\gamma_0$ , connects  $-1$  to  $\xi_0$ . It is formed by creating and increasing a droplet of 0-spins in a sea of  $-1$ -spins.

Let  $\gamma_0 = (-1 = \eta_0, \dots, \eta_N = \xi_0)$ , where

- $N = 2(L-3)$ ,
- $\eta_1$  is a  $2 \times 2$  square of 0-spins in a background of negative spins,
- For  $k < N-1$ ,  $\eta_{k+1}$  is obtained from  $\eta_k$  adding a line of 0-spins to transform a  $j \times j$ -square of 0-spins into a  $(j+1) \times j$ -square of 0-spins, or to transform  $(j+1) \times j$ -square of 0-spins into a  $j \times j$ -square of 0-spins.

Note that  $\eta_N$  is obtained from  $\eta_{N-1}$  transforming a  $(L-2) \times (L-2)$  rectangle into a  $L \times (L-2)$  band.

Let  $\xi_1 \in \mathcal{S}$  be the configuration formed by a  $2 \times 2$  square of  $+1$ -spins in a background of 0-spins. The last piece of the path, denoted by  $\gamma_1$ , connects  $\xi_0$  to  $+1$  and is constructed in a similar way as  $\gamma_0$  so that  $\gamma_1 = (\xi_1 = \zeta_0, \dots, \zeta_N = +1)$ . Note that the length of  $\gamma_1$  is the same as the one of  $\gamma_0$ .

Denote by  $q(\eta, \xi)$  the jump probabilities of the trace of  $\sigma(t)$  on  $\mathcal{S}$ :  $q(\eta, \xi) = \mathbb{P}_\eta[H_\xi = H_{\mathcal{S} \setminus \{\eta\}}]$ . Let

$$q(\gamma_0) = \prod_{k=0}^{N-1} q(\eta_k, \eta_{k+1}), \quad q(\gamma_1) = \prod_{k=0}^{N-1} q(\zeta_k, \zeta_{k+1}),$$

so that

$$\mathbb{P}_{-1}[H_{+1} < H_{\{-1,0\}}^+] \geq q(\gamma_0) q(\xi_0, \xi_1) q(\gamma_1). \quad (6.2)$$

We estimate the three terms on the right hand side.

**Assertion 6.A.** *There exists a positive constant  $c_0$ , independent of  $\beta$ , such that*

$$q(\gamma_0) \geq c_0 e^{-\beta\{4(n_0-1)-[n_0(n_0+1)-2]h\}}.$$

*Proof.* By the arguments presented in the proof of Assertion 4.H, there exists a positive constant  $c_0$ , independent of  $\beta$ , such that  $q(\eta_k, \eta_{k+1}) \geq c_0$  if  $k \geq 2n_0 - 3$ . Thus,

$$q(\gamma_0) \geq c_0 \prod_{k=0}^{2(n_0-2)} q(\eta_k, \eta_{k+1}),$$

and  $\eta_{2n_0-3}$  is a  $(n_0+1) \times (n_0+1)$  square of 0 spins in a sea of  $-1$ -spins.

Denote by  $\lambda_{\mathcal{S}}$  the holding rates of the trace of  $\sigma(t)$  on  $\mathcal{S}$ , by  $\mu_{\mathcal{S}}$  the invariant probability measure, and let  $M_{\mathcal{S}}(\eta) = \lambda_{\mathcal{S}}(\eta) \mu_{\mathcal{S}}(\eta)$ . The measure  $M_{\mathcal{S}}$  is reversible for the discrete-time chain which jumps from  $\eta$  to  $\xi$  with probability  $q(\eta, \xi)$ .

By the proof of Assertion 4.H, there exists a positive constant  $c_0$ , independent of  $\beta$ , such that  $q(\eta_{k+1}, \eta_k) \geq c_0$  if  $k < 2(n_0 - 2)$ . Thus, multiplying and dividing by  $M_S(-\mathbf{1})$ , by reversibility

$$\begin{aligned} \prod_{k=0}^{2(n_0-2)} q(\eta_k, \eta_{k+1}) &= \frac{M_S(\eta_{2(n_0-2)})}{M_S(-\mathbf{1})} \prod_{k=0}^{2n_0-5} q(\eta_{k+1}, \eta_k) q(\eta_{2(n_0-2)}, \eta_{2n_0-3}) \\ &\geq c_0 \frac{M_S(\eta_{2(n_0-2)})}{M_S(-\mathbf{1})} q(\eta_{2(n_0-2)}, \eta_{2n_0-3}), \end{aligned}$$

where the configuration  $\eta_{2(n_0-2)}$  is a  $(n_0 + 1) \times n_0$  rectangle of 0-spins.

Recall that  $M_S(\eta) = \mu_S(\eta) \lambda_S(\eta)$ . Since  $\mu_S(\eta) = \mu_\beta(\eta) / \mu_\beta(S)$ , by [2, Proposition 6.1], for any  $\eta \in S$ ,

$$M_S(\eta) = \frac{\mu_\beta(\eta)}{\mu_\beta(S)} \lambda_\beta(\eta) \mathbb{P}_\eta[H_{S \setminus \{\eta\}} < H_\eta^+] = \frac{\text{cap}(\eta, S \setminus \{\eta\})}{\mu_\beta(S)}. \quad (6.3)$$

We claim that

$$M_S(\eta_{2(n_0-2)}) q(\eta_{2(n_0-2)}, \eta_{2n_0-3}) \geq c_0 \mu_S(\eta_{2(n_0-2)}) e^{-\beta(2-h)}. \quad (6.4)$$

To keep notation simple, let  $\eta = \eta_{2(n_0-2)}$ ,  $\xi = \eta_{2n_0-3}$ . By definition of  $q$  and by (3.1), the jump probability appearing on the left hand side is equal to

$$\mathbb{P}_\eta[H_\xi = H_{S \setminus \{\eta\}}] = \frac{\mu_\beta(\eta) \lambda_\beta(\eta) \mathbb{P}_\eta[H_\xi = H_S^+]}{\text{cap}(\eta, S \setminus \{\eta\})}.$$

The denominator cancels the numerator in (6.3). On the other hand, to reach  $\xi$  from  $\eta$  without returning to  $\eta$ , the simplest way consists in creating a 0-spin attached to the longer side of the rectangle and to build a line of 0-spins from this first one. Only the first creation has a cost which vanishes as  $\beta \uparrow \infty$ . Hence,  $\lambda_\beta(\eta) \mathbb{P}_\eta[H_\xi = H_S^+] \geq c_0 R_\beta(\eta, \eta')$  where  $\eta'$  is a critical configuration in  $\mathfrak{R}^l$ . This completes the proof of (6.4) since  $R_\beta(\eta, \eta') = \exp\{-\beta(2-h)\}$ .

It remains to estimate  $M_S(-\mathbf{1})$ . Recall (4.4). Since  $\xi_{-\mathbf{1}, S \setminus \{-\mathbf{1}\}}$  is the configuration with three 0-spins included in a  $2 \times 2$  square,  $\text{cap}(-\mathbf{1}, S \setminus \{-\mathbf{1}\}) \leq C_0 \exp\{-\beta[8-3h]\} \mu_\beta(-\mathbf{1})$ . Hence, by (6.3),

$$M_S(-\mathbf{1}) \leq C_0 e^{-\beta(8-3h)} \mu_S(-\mathbf{1}). \quad (6.5)$$

Putting together all previous estimates, we obtain that

$$q(\gamma_0) \geq c_0 \frac{\mu_\beta(\eta_{2(n_0-2)})}{\mu_\beta(-\mathbf{1})} e^{-\beta(2-h)} e^{\beta(8-3h)},$$

which completes the proof of the assertion in view of the definition of  $\eta_{2(n_0-2)}$ .  $\square$

Next result is proved similarly.

**Assertion 6.B.** *There exists a positive constant  $c_0$ , independent of  $\beta$ , such that*

$$q(\gamma_1) \geq c_0 e^{-\beta\{4(n_0-1)-[n_0(n_0+1)-2]h\}}.$$

We turn to the probability  $q(\xi_0, \xi_1)$ . Recall that  $\xi_0$  is the configuration formed by a  $L \times (L-2)$  band of 0-spins and a  $L \times 2$  band of  $-1$  spins, and that  $\xi_1$  is the configuration formed by a  $2 \times 2$  square of  $+1$ -spins in a background of 0-spins.

**Assertion 6.C.** *There exists a positive constant  $c_0$ , independent of  $\beta$ , such that*

$$q(\xi_0, \xi_1) \geq c_0 e^{-2\beta[2-h]}.$$

*Proof.* By definition of  $q$  and by (3.1),

$$q(\xi_0, \xi_1) = \mathbb{P}_{\xi_0}[H_{\xi_1} = H_{\mathcal{S} \setminus \{\xi_0\}}] = \frac{\mu_\beta(\xi_0) \lambda_\beta(\xi_0) \mathbb{P}_{\xi_0}[H_{\xi_1} = H_{\mathcal{S}}^+]}{\text{cap}(\xi_0, \mathcal{S} \setminus \{\xi_0\})}.$$

We claim that

$$\mathbb{P}_{\xi_0}[H_{\xi_1} = H_{\mathcal{S}}^+] \geq c_0 e^{-2\beta[2-h]}. \quad (6.6)$$

To estimate this probability, we propose a path  $\gamma_3$  from  $\xi_0$  to  $\xi_1$  which avoids  $\mathcal{S}$ . The path consists in filling the  $-1$ -spins with  $0$ -spins, until one  $-1$ -spin is left. At this point, to avoid the configuration  $\mathbf{0}$ , we switch this  $-1$ -spin to  $+1$ . To complete the path we create a  $2 \times 2$  square of  $+1$ -spins from the first  $+1$ -spin, as in the optimal path from  $\mathbf{0}$  to  $\xi_1$ .

Hence,  $\gamma_3$  as length  $2L + 3$ . Denote this path by  $\gamma_3 = (\xi_0 = \eta'_0, \eta'_1, \dots, \eta'_{2L+3} = \xi_1)$ . From  $\eta'_0$  to  $\eta'_{2L-2}$  the next configurations is obtained by flipping a  $-1$  spin to a  $0$ -spin as in an optimal path from  $\xi_0$  to  $\mathbf{0}$ . In this piece of the path, all jumps have a probability bounded below by a positive constant. Therefore, there exists a positive constant  $c_0$ , independent of  $\beta$ , such that

$$\mathbb{P}_{\xi_0}[H_{\xi_1} = H_{\mathcal{S}}^+] \geq \prod_{j=0}^{2L+2} p_\beta(\eta'_j, \eta'_{j+1}) \geq c_0 \prod_{j=2L-1}^{2L+2} p_\beta(\eta'_j, \eta'_{j+1}).$$

The first and the last probabilities in this product,  $p_\beta(\eta'_{2L-1}, \eta'_{2L})$  and  $p_\beta(\eta'_{2L+2}, \eta'_{2L+3})$ , are also bounded below by a positive constant. The other ones can be estimated easily, proving (6.6).

By (4.4),  $\text{cap}(\xi_0, \mathcal{S} \setminus \{\xi_0\})/\mu_\beta(\xi_0)$  is bounded above by  $C_0 \exp\{-\beta[2-h]\}$ , while an elementary computation shows that  $\lambda_\beta(\xi_0)$  is bounded below by  $c_0 \exp\{-\beta[2-h]\}$ . This completes the proof of the assertion.  $\square$

By (6.2) and Assertions 6.A, 6.B and 6.C,

$$\mathbb{P}_{-1}[H_{+1} < H_{\{-1, \mathbf{0}\}}^+] \geq c_0 e^{-2\beta\{2(2n_0-1) - [n_0(n_0+1)-1]h\}}. \quad (6.7)$$

*Proof of Lemma 6.1.* Since  $\lambda(-1) \geq c_0 e^{-\beta[4-h]}$ , by (6.1), (6.7),

$$\mathbb{P}_{-1}[H_{+1} < H_0] \geq c_0 \frac{\mu_\beta(-1)}{\text{cap}(-1, \{-1, \mathbf{0}\})} e^{-\beta\{8n_0 - [2n_0(n_0+1)-1]h\}}.$$

Therefore, by Proposition 2.3,

$$\frac{\mu_\beta(+1)}{\mu_\beta(\mathbf{0})} \mathbb{P}_{-1}[H_{+1} < H_0] \geq c_0 e^{\beta L^2} e^{-\beta\{4(n_0-1) - [n_0(n_0+1)-2]h\}}.$$

It remains to show that  $L^2 > 4(n_0-1) - [n_0(n_0+1)-2]h$ . By definition of  $n_0$ ,  $n_0 h > 2-h$ , so that  $4(n_0-1) - [n_0(n_0+1)-2]h \leq 2n_0 - 6 + hn_0 + 3h$ . As  $hn_0 < 2$  and  $h < 1$ , this expression is less than or equal to  $2n_0$ . This expression is smaller than  $L^2$  because  $L \geq 2$  and  $L > n_0$ .  $\square$

## 7. PROOF OF THEOREM 2.5

The statement of Theorem 2.5 follows from Propositions 7.1 and 7.4 below and from Theorem 5.1 in [18]. We start deriving some consequences of the assumption (2.10). Clearly, it follows from (2.10) and from (4.4) that for all  $\eta \in \mathcal{M}$  and  $\sigma \in \mathcal{V}_\eta$ ,  $\sigma \neq \eta$ ,

$$\frac{\mu_\beta(\eta)}{\text{cap}(\sigma, \eta)} \prec \theta_\beta. \quad (7.1)$$

**Assertion 7.A.** *For all  $\eta \in \mathcal{M}$  and  $\sigma \in \mathcal{V}_\eta$ ,  $\sigma \neq \eta$ ,*

$$\mu_\beta(\sigma) \prec \mu_\beta(\eta), \quad \text{cap}(\eta, \mathcal{M} \setminus \{\eta\}) \prec \text{cap}(\sigma, \eta).$$

*Proof.* The first bound is a straightforward consequence of the hypothesis  $\mathbb{H}(\sigma) > \mathbb{H}(\eta)$ . In view of (4.4), to prove the second bound we have to show that  $\mathbb{H}(\sigma, \eta) < \mathbb{H}(\eta, \mathcal{M} \setminus \{\eta\})$ . The case  $\eta = -\mathbf{1}$  is a consequence of the second hypothesis in (2.10), as the case  $\eta = \mathbf{0}$  if one recalls (2.11). It remains to consider the case  $\eta = +\mathbf{1}$ . By condition (2.10) and (2.11),

$$\mathbb{H}(\sigma, +\mathbf{1}) < \mathbb{H}(+\mathbf{1}) + \mathbb{H}(\mathbf{0}, \{-\mathbf{1}, +\mathbf{1}\}) - \mathbb{H}(\mathbf{0}) < \mathbb{H}(\mathbf{0}, \{-\mathbf{1}, +\mathbf{1}\}).$$

By Assertion 5.E and by (4.4),  $\mathbb{H}(\mathbf{0}, \{-\mathbf{1}, +\mathbf{1}\}) = \mathbb{H}(+\mathbf{1}, \{-\mathbf{1}, \mathbf{0}\})$ . Therefore,

$$\mathbb{H}(\sigma, +\mathbf{1}) < \mathbb{H}(+\mathbf{1}, \{-\mathbf{1}, \mathbf{0}\}),$$

which proves the second claim of the assertion.  $\square$

**Assertion 7.B.** *For all  $\eta \neq \xi \in \mathcal{M}$  and for all  $\sigma \in \mathcal{V}_\eta$ ,  $\sigma' \in \mathcal{V}_\xi$ ,*

$$\text{cap}(\sigma, \sigma') \approx \text{cap}(\eta, \xi).$$

*Proof.* Fix  $\eta \neq \xi \in \mathcal{M}$  and  $\sigma \in \mathcal{V}_\eta$ ,  $\sigma' \in \mathcal{V}_\xi$ . We need to prove that  $\mathbb{H}(\eta, \xi) = \mathbb{H}(\sigma, \sigma')$ . On the one hand, by definition,  $\mathbb{H}(\sigma, \sigma') \leq \max\{\mathbb{H}(\sigma, \eta), \mathbb{H}(\eta, \eta'), \mathbb{H}(\eta', \sigma')\}$ . By the proof of Assertion 7.A,  $\mathbb{H}(\sigma, \eta) < \mathbb{H}(\eta, \mathcal{M} \setminus \{\eta\})$ , with a similar inequality replacing  $\sigma, \eta$  by  $\sigma', \eta'$ , respectively. Since  $\mathbb{H}(\mathcal{A}, \mathcal{B})$  is decreasing in each variable,  $\mathbb{H}(\eta, \mathcal{M} \setminus \{\eta\})$  and  $\mathbb{H}(\eta', \mathcal{M} \setminus \{\eta'\})$  are less than or equal to  $\mathbb{H}(\eta, \eta')$ , which shows that  $\mathbb{H}(\sigma, \sigma') \leq \mathbb{H}(\eta, \eta')$ .

Conversely,  $\mathbb{H}(\eta, \eta') \leq \max\{\mathbb{H}(\eta, \sigma), \mathbb{H}(\sigma, \sigma'), \mathbb{H}(\sigma', \eta')\}$ . By the previous paragraph,  $\mathbb{H}(\eta, \sigma) < \mathbb{H}(\eta, \eta')$  and  $\mathbb{H}(\sigma', \eta') < \mathbb{H}(\eta, \eta')$  so that  $\mathbb{H}(\eta, \eta') \leq \mathbb{H}(\sigma, \sigma')$ . This completes the proof of the assertion.  $\square$

We conclude this preamble with two simple remarks. Fix  $\eta \in \mathcal{M}$  and  $\sigma \in \mathcal{V}_\eta$ . By (3.4) and Assertion 7.B,

$$\text{cap}(\sigma, \cup_{\xi \neq \eta} \mathcal{V}_\xi) \approx \max_{\sigma' \in \cup_{\xi \neq \eta} \mathcal{V}_\xi} \text{cap}(\sigma, \sigma') \approx \max_{\xi \in \mathcal{M} \setminus \{\eta\}} \text{cap}(\eta, \xi).$$

Applying (3.4) once more, we conclude that

$$\text{cap}(\sigma, \cup_{\xi \neq \eta} \mathcal{V}_\xi) \approx \text{cap}(\eta, \mathcal{M} \setminus \{\eta\}). \quad (7.2)$$

In particular, by Assertion 7.A,

$$\lim_{\beta \rightarrow \infty} \frac{\text{cap}(\sigma, \cup_{\xi \neq \eta} \mathcal{V}_\xi)}{\text{cap}(\sigma, \eta)} = 0. \quad (7.3)$$

Denote by  $\sigma^\mathcal{A}(t)$ ,  $\mathcal{A} \subset \Omega$ , the trace of  $\sigma(t)$  on  $\mathcal{A}$ . By [2, Proposition 6.1],  $\sigma^\mathcal{A}(t)$  is a continuous-time Markov chain. Moreover, for  $\mathcal{B} \subset \mathcal{A}$ ,  $\sigma^\mathcal{B}(t)$  is the trace of  $\sigma^\mathcal{A}(t)$  on  $\mathcal{B}$ . When  $\mathcal{A} = \mathcal{M}$ , we represent  $\sigma^\mathcal{A}(t)$  by  $\eta(t)$ . Denote  $R_\beta^\mathcal{A}(\sigma, \sigma')$ ,  $\sigma \neq \sigma' \in \mathcal{A}$ , the jump rates of the Markov chain  $\sigma^\mathcal{A}(t)$ .

Recall the definition of the map  $\pi : \mathcal{M} \rightarrow \{-1, 0, 1\}$ , introduced just before the statement of Theorem 2.5. Denote by  $\psi = \psi_\mathcal{V} : \mathcal{V} \rightarrow \{-1, 0, 1\}$  the projections defined by  $\psi(\sigma) = \pi(\eta)$  if  $\sigma \in \mathcal{V}_\eta$ :

$$\psi(\sigma) = \sum_{\eta \in \mathcal{M}} \pi(\eta) \mathbf{1}\{\sigma \in \mathcal{V}_\eta\}.$$

Recall also the definition of the time-scale  $\theta_\beta$  introduced in (2.6).

**Proposition 7.1.** *As  $\beta \uparrow \infty$ , the speeded-up, hidden Markov chain  $\psi(\sigma^\vee(\theta_\beta t))$  converges to the continuous-time Markov chain  $X(t)$  introduced in Theorem 2.5.*

We first prove Proposition 7.1 in the case where the wells  $\mathcal{V}_\eta$  are singletons:  $\mathcal{V}_\eta = \{\eta\}$ . In this case,  $\psi$  is a bijection, and  $\psi(\eta(t))$  is a Markov chain on  $\{-1, 0, 1\}$ .

**Lemma 7.2.** *As  $\beta \uparrow \infty$ , the speeded-up chain  $\eta(\theta_\beta t)$  converges to the continuous-time Markov chain on  $\mathcal{M}$  in which  $+1$  is an absorbing state, and whose jump rates  $\mathbf{r}(\eta, \xi)$ , are given by*

$$\mathbf{r}(-1, 0) = \mathbf{r}(0, +1) = 1, \quad \mathbf{r}(-1, +1) = \mathbf{r}(0, -1) = 0.$$

*Proof.* Denote by  $r_\beta(\eta, \xi)$  the jump rates of the chain  $\eta(\theta_\beta t)$ . It is enough to prove that

$$\lim_{\beta \rightarrow \infty} r_\beta(\eta, \xi) = \mathbf{r}(\eta, \xi) \quad (7.4)$$

for all  $\eta \neq \xi \in \mathcal{M}$ .

By [2, Proposition 6.1], the jump rates  $r_\beta(\eta, \xi)$ ,  $\eta \neq \xi \in \mathcal{M}$ , of the Markov chain  $\eta_\beta(t)$  are given by

$$r_\beta(\eta, \xi) = \theta_\beta \lambda(\eta) \mathbb{P}_\eta[H_\xi = H_\mathcal{M}^+].$$

Dividing and multiplying the previous expression by  $\mathbb{P}_\eta[H_{\mathcal{M} \setminus \{\eta\}} < H_\eta^+]$ , in view of [2, Lemma 6.6] and of (3.1), we obtain that

$$r_\beta(\eta, \xi) = \frac{\theta_\beta}{\mu_\beta(\eta)} \text{cap}(\eta, \mathcal{M} \setminus \{\eta\}) \mathbb{P}_\eta[H_\xi < H_{\mathcal{M} \setminus \{\eta, \xi\}}].$$

For  $\eta = +1$  and  $\xi = -1, 0$ , by Assertion 5.E and by Proposition 2.3,

$$\lim_{\beta \rightarrow \infty} r_\beta(+1, \xi) \leq \lim_{\beta \rightarrow \infty} \frac{\theta_\beta}{\mu_\beta(+1)} \text{cap}(+1, \mathcal{M} \setminus \{+1\}) = \lim_{\beta \rightarrow \infty} \frac{\mu_\beta(0)}{\mu_\beta(+1)} = 0.$$

On the other hand, by Proposition 2.3,

$$\lim_{\beta \rightarrow \infty} \frac{\theta_\beta}{\mu_\beta(0)} \text{cap}(0, \mathcal{M} \setminus \{0\}) = 1,$$

while  $\theta_\beta \text{cap}(-1, \mathcal{M} \setminus \{-1\}) / \mu_\beta(0) = 1$ . Furthermore, by Proposition 2.1 and Assertion 5.C,

$$\lim_{\beta \rightarrow \infty} \mathbb{P}_{-1}[H_{+1} < H_0] = \lim_{\beta \rightarrow \infty} \mathbb{P}_0[H_{-1} < H_{+1}] = 0.$$

This yields (7.4) and completes the proof of the lemma.  $\square$

Denote by  $\mathbb{P}_\sigma^\vee$ ,  $\sigma \in \mathcal{V}$ , the probability measure on the path space  $D(\mathbb{R}_+, \mathcal{V})$  induced by the Markov chain  $\sigma^\vee(t)$  starting from  $\sigma$ . Expectation with respect to  $\mathbb{P}_\sigma^\vee$  is represented by  $\mathbb{E}_\sigma^\vee$ . Clearly, for any disjoint subsets  $\mathcal{A}, \mathcal{B}$  of  $\mathcal{V}$ ,

$$\mathbb{P}_\sigma^\vee[H_\mathcal{A} < H_\mathcal{B}] = \mathbb{P}_\sigma[H_\mathcal{A} < H_\mathcal{B}]. \quad (7.5)$$

The hitting time of a subset  $\mathcal{A}$  of  $\mathcal{V}$  by the trace chain  $\sigma^\vee$  can be represented in terms of the original chain  $\sigma(t)$ . Under  $\mathbb{P}_\sigma$ ,

$$H_\mathcal{A}^\vee = \inf\{t > 0 : \sigma^\vee(t) \in \mathcal{A}\} = \int_0^{H_\mathcal{A}} \mathbf{1}\{\sigma(t) \in \mathcal{V}\} dt. \quad (7.6)$$

Let

$$\check{\mathcal{V}}(\eta) = \check{\mathcal{V}}_\eta = \bigcup_{\zeta \neq \eta} \mathcal{V}_\zeta.$$



Denote by  $\{T_j : j \geq 0\}$  the jump times of the hidden chain  $\psi(\sigma^\vee(t))$ :

$$T_0 = 0, \quad T_{j+1} = \inf\{t \geq T_j : \sigma^\vee(t) \in \check{\mathcal{V}}(\sigma^\vee(T_j))\}, \quad j \geq 0,$$

Similarly, denote by  $\{\tau_j : j \geq 0\}$  the successive jump times of the chain  $\eta(t)$ .

**Lemma 7.3.** *Fix  $\sigma \in \mathcal{V}_{-1}$ . There exists a sequence  $\epsilon_\beta \rightarrow 0$  such that for  $j = 1, 2$*

$$\lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma^\vee[|\tau_j - T_j| \geq \theta_\beta \epsilon_\beta] = 0, \quad \lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma^\vee[T_3 - T_2 \leq \theta_\beta \epsilon_\beta^{-1}] = 0. \quad (7.7)$$

Moreover,

$$\lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma^\vee[\sigma(T_1) \notin \mathcal{V}_0] = 0, \quad \lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma^\vee[\sigma(T_2) \notin \mathcal{V}_{+1}] = 0. \quad (7.8)$$

*Proof.* Fix a configuration  $\sigma \in \mathcal{V}_{-1}$ . By (7.5), (3.3) and (7.3),

$$\limsup_{\beta \rightarrow \infty} \mathbb{P}_\sigma^\vee[H_{\check{\mathcal{V}}(-1)} < H_{-1}] \leq \lim_{\beta \rightarrow \infty} \frac{\text{cap}(\sigma, \check{\mathcal{V}}(-1))}{\text{cap}(\sigma, -1)} = 0. \quad (7.9)$$

On the other hand, under  $\mathbb{P}_\sigma^\vee$ ,

$$\tau_1 = \int_{H_{-1}}^{H_{0,+1}} \mathbf{1}\{\sigma(s) = -1\} ds.$$

Hence, under  $\mathbb{P}_\sigma^\vee$  and on the event  $\{H_{-1} < H_{\check{\mathcal{V}}(-1)}\}$  we have that

$$\begin{aligned} T_1 &= H_{-1} + \int_{H_{-1}}^{H(\check{\mathcal{V}}(-1))} \{\mathbf{1}\{\sigma(s) = -1\} + \mathbf{1}\{\sigma(s) \neq -1\}\} ds \\ &= \tau_1 + H_{-1} + \int_{H_{-1}}^{H(\check{\mathcal{V}}(-1))} \mathbf{1}\{\sigma(s) \neq -1\} ds - \int_{H(\check{\mathcal{V}}(-1))}^{H_{0,+1}} \mathbf{1}\{\sigma(s) = -1\} ds. \end{aligned}$$

It remains to estimate the last three terms.

To bound the first term, by (7.5), (3.3), and (7.3),

$$\limsup_{\beta \rightarrow \infty} \mathbb{P}_\sigma^\vee[H_{\check{\mathcal{V}}(-1)} < H_{-1}] = \limsup_{\beta \rightarrow \infty} \mathbb{P}_\sigma[H_{\check{\mathcal{V}}(-1)} < H_{-1}] \leq \lim_{\beta \rightarrow \infty} \frac{\text{cap}(\sigma, \check{\mathcal{V}}(-1))}{\text{cap}(\sigma, -1)} = 0.$$

Hence, to prove that  $\mathbb{P}_\sigma^\vee[H_{-1} > \theta_\beta \epsilon_\beta] \rightarrow 0$ , it is enough to show that

$$\lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma^\vee[H_{\check{\mathcal{V}}(-1) \cup \{-1\}} > \theta_\beta \epsilon_\beta] = 0.$$

By Tchebycheff inequality, by Lemma 6.9 and Proposition 6.10 in [2], and by (7.5), the previous probability is less than or equal to

$$\frac{1}{\theta_\beta \epsilon_\beta} \frac{1}{\text{cap}(\sigma, \check{\mathcal{V}}(-1) \cup \{-1\})} \sum_{\eta \in \mathcal{V}} \mu_\beta(\eta) \mathbb{P}_\eta[H_\sigma < H_{\check{\mathcal{V}}(-1) \cup \{-1\}}].$$

By definition of  $\theta_\beta$ , since the capacity is monotone, and since we may restrict the sum to  $\mathcal{V}_{-1}$ , the previous expression is less than or equal to

$$\frac{1}{\epsilon_\beta} \frac{\text{cap}(-1, \{\mathbf{0}, +1\})}{\text{cap}(\sigma, -1)} \frac{\mu_\beta(\mathcal{V}_{-1})}{\mu_\beta(-1)}.$$

By Assertion 7.A,  $\mu_\beta(\mathcal{V}_{-1})/\mu_\beta(-1)$  is bounded and  $\text{cap}(-1, \{\mathbf{0}, +1\})/\text{cap}(\sigma, -1)$  vanishes as  $\beta \uparrow \infty$ . Hence, the previous expression converges to 0 for every sequence  $\epsilon_\beta$  which does not decrease too fast.

We turn to the second term of the decomposition of  $T_1$ . By the strong Markov property and by (7.6), we need to estimate,

$$\begin{aligned} & \mathbb{P}_{-1}^{\mathcal{V}} \left[ \int_0^{H(\check{\mathcal{V}}_{-1})} \mathbf{1}\{\sigma(s) \neq -1\} ds > \theta_\beta \epsilon_\beta \right] \\ &= \mathbb{P}_{-1} \left[ \int_0^{H(\check{\mathcal{V}}_{-1})} \mathbf{1}\{\sigma(s) \in \mathcal{V} \setminus \{-1\}\} ds > \theta_\beta \epsilon_\beta \right]. \end{aligned}$$

By Tchebycheff inequality and by [2, Proposition 6.10], the previous probability is less than or equal to

$$\frac{1}{\theta_\beta \epsilon_\beta} \frac{1}{\text{cap}(-1, \check{\mathcal{V}}_{-1})} \sum_{\eta \in \mathcal{V} \setminus \{-1\}} \mu_\beta(\eta) \mathbb{P}_\eta[H_{-1} < H_{\check{\mathcal{V}}_{-1}}].$$

By (7.2),  $\text{cap}(-1, \check{\mathcal{V}}_{-1}) \approx \text{cap}(-1, \{\mathbf{0}, +1\})$ . Hence, by definition of  $\theta_\beta$ , and since the sum can be restricted to the set  $\mathcal{V}_{-1} \setminus \{-1\}$ , the previous expression is less than or equal to

$$\frac{C_0}{\epsilon_\beta} \frac{1}{\mu_\beta(-1)} \mu_\beta(\mathcal{V}_{-1} \setminus \{-1\})$$

for some finite constant  $C_0$ . By Assertion 7.A, the ratio of the measures vanishes as  $\beta \uparrow \infty$ . In particular, the previous expression converges to 0 as  $\beta \uparrow \infty$  if  $\epsilon_\beta$  does not decrease too fast.

The third term in the decomposition of  $T_1$  is absolutely bounded by  $H_{\mathbf{0},+1} - H(\check{\mathcal{V}}_\eta)$  and can be handled as the first one. This proves the first assertion of (7.7) for  $j = 1$ .

In a similar way one proves that  $T_2 - T_1$  is close to  $\tau_2 - \tau_1$ . The first assertion of (7.7) for  $j = 2$  follows from this result and from the bound for  $T_1 - \tau_1$ . The details are left to the reader.

We turn to the proof of the first assertion in (7.8). Since  $\check{\mathcal{V}}_{-1} = \mathcal{V}_{\mathbf{0}} \cup \mathcal{V}_{+1}$ ,

$$\mathbb{P}_\sigma^{\mathcal{V}}[\sigma(T_1) \notin \mathcal{V}_{\mathbf{0}}] = \mathbb{P}_\sigma[\sigma(H) \in \mathcal{V}_{+1}],$$

where  $H = H(\check{\mathcal{V}}_{-1})$ . We may rewrite the previous probability as

$$\mathbb{P}_\sigma[\sigma(H) \in \mathcal{V}_{+1}, H_{\mathbf{0}} < H_{+1}] + \mathbb{P}_\sigma[\sigma(H) \in \mathcal{V}_{+1}, H_{\mathbf{0}} > H_{+1}].$$

Both expression vanishes as  $\beta \uparrow \infty$ . The second one is bounded by  $\mathbb{P}_\sigma[H_{+1} < H_{\mathbf{0}}]$ , which vanishes by Proposition 2.1. Since  $H < \min\{H_{\mathbf{0}}, H_{+1}\}$ , by the strong Markov property, the first term is less than or equal to

$$\max_{\sigma' \in \mathcal{V}_{+1}} \mathbb{P}_{\sigma'}[H_{\mathbf{0}} < H_{+1}].$$

This expression converges to 0 as  $\beta \uparrow \infty$  because  $\mathcal{V}_{+1}$  is contained in the basin of attraction of  $+1$ . The proof of the second assertion in (7.8) is similar and left to the reader.

We finally examine the third assertion of (7.7). By the second assertion of (7.8), it is enough to prove that

$$\lim_{\beta \rightarrow \infty} \mathbb{P}_\sigma^{\mathcal{V}}[T_3 - T_2 \leq \theta_\beta \epsilon_\beta^{-1}, \sigma(T_2) \in \mathcal{V}_{+1}] = 0.$$

By the strong Markov property, this limit holds if

$$\lim_{\beta \rightarrow \infty} \max_{\sigma' \in \mathcal{V}_{+1}} \mathbb{P}_{\sigma'}^{\mathcal{V}}[T_1 \leq \theta_\beta \epsilon_\beta^{-1}] = 0.$$

Since  $\mathcal{V}_{+1}$  is contained in the basin of attraction of  $+1$ , it is enough to show that

$$\lim_{\beta \rightarrow \infty} \max_{\sigma' \in \mathcal{V}_{+1}} \mathbb{P}_{\sigma'}^{\mathcal{V}}[T_1 \leq \theta_{\beta} \epsilon_{\beta}^{-1}, H_{+1} < T_1] = 0.$$

On the event  $\{H_{+1} < T_1\}$ ,  $\{T_1 \leq \theta_{\beta} \epsilon_{\beta}^{-1}\} \subset \{T_1 \circ \theta_{H_{+1}} \leq \theta_{\beta} \epsilon_{\beta}^{-1}\}$ , where  $\{\theta_t : t \geq 0\}$  represents the semigroup of time translations. In particular, by the strong Markov property, we just need to show that

$$\lim_{\beta \rightarrow \infty} \mathbb{P}_{+1}^{\mathcal{V}}[T_1 \leq \theta_{\beta} \epsilon_{\beta}^{-1}] = 0.$$

Let  $\{\epsilon_j : j \geq 1\}$  be the length of the sojourn times at  $+1$ . Hence,  $\{\epsilon_j : j \geq 1\}$  is a sequence of i.i.d. exponential random variables with parameter  $\lambda(+1)$ . Denote by  $\mathcal{A}$  the set of configurations with at least  $n_0(n_0 + 1)$  sites with spins not equal to  $+1$ . Each time the process leaves the state  $+1$  it attempts to reach  $\mathcal{A}$  before it returns to  $+1$ . Let  $\delta$  be the probability of success:

$$\delta = \mathbb{P}_{+1}^{\mathcal{V}}[H_{\mathcal{A}} < H_{+1}^+].$$

Let  $N \geq 1$  be the number of attempts up to the first success so that  $\sum_{1 \leq j \leq N} \epsilon_j$  represents the total time the process  $\sigma(t)$  remained at  $+1$  before it reached  $\mathcal{A}$ . It is clear that under  $\mathbb{P}_{+1}^{\mathcal{V}}$ ,

$$\sum_{j=1}^N \epsilon_j \leq T_1,$$

and that  $N$  is a geometric random variable of parameter  $\delta$  independent of the sequence  $\{\epsilon_j : j \geq 1\}$ . In view of the previous inequality, it is enough to prove that

$$\lim_{\beta \rightarrow \infty} \mathbb{P}_{+1}^{\mathcal{V}}\left[\sum_{j=1}^N \epsilon_j \leq \theta_{\beta} \epsilon_{\beta}^{-1}\right] = 0.$$

The previous probability is less than or equal to

$$\lambda(+1) \theta_{\beta} \epsilon_{\beta}^{-1} \mathbb{P}_{+1}^{\mathcal{V}}[H_{\mathcal{A}} < H_{+1}^+] = \frac{1}{\epsilon_{\beta}} \frac{\theta_{\beta}}{\mu_{\beta}(+1)} \text{cap}(\mathcal{A}, +1).$$

Since  $\theta_{\beta} \text{cap}(\mathcal{A}, +1) / \mu_{\beta}(+1) \prec 1$ , the previous expression vanishes if  $\epsilon_{\beta}$  does not decrease too fast to 0. This completes the proof of the lemma.  $\square$

*Proof of Proposition 7.1.* The assertion of the proposition is a straightforward consequence of Lemmas 7.2 and 7.3.

Fix  $\sigma \in \mathcal{V}_{-1}$  and recall the notation introduced in Lemma 7.3. Let  $\mathcal{A} = \{\sigma(T_1) \in \mathcal{V}_0\} \cap \{\sigma(T_2) \in \mathcal{V}_{+1}\}$ . By (7.8),  $\mathbb{P}_{\sigma}^{\mathcal{V}}[\mathcal{A}^c] \rightarrow 0$ . On the set  $\mathcal{A}$ ,

$$\psi(\sigma^{\mathcal{V}}(\theta_{\beta} t)) = -\mathbf{1}\{t < T_1/\theta_{\beta}\} + \mathbf{1}\{T_2/\theta_{\beta} \leq t < T_3/\theta_{\beta}\}$$

By Lemma 7.2,  $(\tau_1/\theta_{\beta}, (\tau_2 - \tau_1)/\theta_{\beta})$  converges to a pair of independent, mean 1, exponential random variables. Hence, by (7.7),  $(T_1/\theta_{\beta}, (T_2 - T_1)/\theta_{\beta}, (T_3 - T_2)/\theta_{\beta})$  converges in distribution to  $(\epsilon_1, \epsilon_2, \infty)$ , where  $(\epsilon_1, \epsilon_2)$  is a pair of independent, mean 1, exponential random variables. This completes the proof.  $\square$

**Lemma 7.4.** *Let  $\Delta = \Omega \setminus \mathcal{V}$ . For all  $\xi \in \mathcal{V}$ ,  $t > 0$ ,*

$$\lim_{\beta \rightarrow \infty} \mathbb{E}_{\xi} \left[ \int_0^t \mathbf{1}\{\sigma(s\theta_{\beta}) \in \Delta\} ds \right] = 0.$$

*Proof.* Fix  $\xi \in \mathcal{V}_{+1}$ . On the one hand, by [2, Proposition 6.10],

$$\frac{1}{\theta_\beta} \mathbb{E}_\xi \left[ \int_0^{H_{+1}} \mathbf{1}\{\sigma(s) \in \Delta\} ds \right] \leq \frac{1}{\theta_\beta} \frac{\mu_\beta(+1)}{\text{cap}(\xi, +1)} \frac{\mu_\beta(\Delta)}{\mu_\beta(+1)}.$$

This expression vanishes as  $\beta \uparrow \infty$  because, by (7.1),  $\mu_\beta(+1)/\text{cap}(\xi, +1) \preceq \theta_\beta$ , and because  $\mu_\beta(\Delta) \prec \mu_\beta(+1)$ , as  $+1$  is the unique ground state.

On the other hand, by the strong Markov property,

$$\frac{1}{\theta_\beta} \mathbb{E}_\xi \left[ \int_{H_{+1}}^{t\theta_\beta} \mathbf{1}\{\sigma(s) \in \Delta\} ds \right] \leq \frac{1}{\theta_\beta} \mathbb{E}_{+1} \left[ \int_0^{t\theta_\beta} \mathbf{1}\{\sigma(s) \in \Delta\} ds \right].$$

Therefore, to prove the lemma for  $\xi \in \mathcal{V}_{+1}$  it is enough to show that

$$\lim_{\beta \rightarrow \infty} \mathbb{E}_{+1} \left[ \int_0^t \mathbf{1}\{\sigma(s\theta_\beta) \in \Delta\} ds \right] = 0.$$

This last assertion follows from Lemma 7.5 below.

Similar arguments permit to reduce the statement of the lemma for  $\xi \in \mathcal{V}_0$  (resp.  $\xi \in \mathcal{V}_{-1}$ ) to the verification that

$$\lim_{\beta \rightarrow \infty} \mathbb{E}_\zeta \left[ \int_0^t \mathbf{1}\{\sigma(s\theta_\beta) \in \Delta\} ds \right] = 0,$$

for  $\zeta = 0$  (resp.  $\zeta = -1$ ), which follows from Lemma 7.5 below.  $\square$

**Lemma 7.5.** *Let  $\Delta^* = \Omega \setminus \mathcal{M}$ . For all  $\xi \in \mathcal{M}$ ,  $t > 0$ ,*

$$\lim_{\beta \rightarrow \infty} \mathbb{E}_\xi \left[ \int_0^t \mathbf{1}\{\sigma(s\theta_\beta) \in \Delta^*\} ds \right] = 0.$$

*Proof.* Consider first the case  $\xi = +1$ . Clearly,

$$\mathbb{E}_{+1} \left[ \int_0^t \mathbf{1}\{\sigma(s\theta_\beta) \in \Delta^*\} ds \right] \leq \frac{1}{\mu_\beta(+1)} \sum_{\sigma \in \Omega} \mu_\beta(\sigma) \mathbb{E}_\sigma \left[ \int_0^t \mathbf{1}\{\sigma(s\theta_\beta) \in \Delta^*\} ds \right].$$

Since  $\mu_\beta$  is the stationary state, the previous expression is equal to

$$\frac{t\mu_\beta(\Delta^*)}{\mu_\beta(+1)},$$

which vanishes as  $\beta \rightarrow \infty$  because  $+1$  is the unique ground state.

We turn to the case  $\xi = 0$ . We first claim that

$$\lim_{\beta \rightarrow \infty} \mathbb{E}_0 \left[ \frac{1}{\theta_\beta} \int_0^{H_{\{-1, +1\}}} \mathbf{1}\{\sigma(s) \in \Delta^*\} ds \right] = 0. \quad (7.10)$$

Indeed, by [2, Proposition 6.10], the previous expectation is equal to

$$\frac{1}{\theta_\beta} \frac{\langle V \mathbf{1}\{\Delta^*\} \rangle_{\mu_\beta}}{\text{cap}(0, \{-1, +1\})} = \frac{\mu_\beta(0)}{\theta_\beta \text{cap}(0, \{-1, +1\})} \frac{\langle V \mathbf{1}\{\Delta^*\} \rangle_{\mu_\beta}}{\mu_\beta(0)},$$

where  $V$  is the harmonic function  $V(\sigma) = \mathbb{P}_\sigma[H_0 < H_{\{-1, +1\}}]$ . By definition of  $\theta_\beta$ , the first fraction on the right hand side is bounded.

It remains to estimate the second fraction, which is equal to

$$\frac{1}{\mu_\beta(0)} \sum_{\sigma \in \Delta_0^*} \mu_\beta(\sigma) \mathbb{P}_\sigma[H_0 < H_{\{-1, +1\}}],$$

where  $\Delta_0^* = \{\sigma \in \Delta^* : \mu_\beta(\sigma) \succeq \mu_\beta(\mathbf{0})\}$ . By (3.3), this sum is less than or equal to

$$\sum_{\sigma \in \Delta_0^*} \frac{\text{cap}(\sigma, \mathbf{0})}{\mu_\beta(\mathbf{0})} \frac{\mu_\beta(\sigma)}{\text{cap}(\sigma, \mathcal{M})}.$$

Each term of this sum vanishes as  $\beta \uparrow \infty$ . Indeed, as  $\sigma$  belongs to  $\Delta_0^*$ , to reach  $\sigma$  from  $\mathbf{0}$  the chain has to escape from the well of  $\mathbf{0}$  so that  $\text{cap}(\sigma, \mathbf{0})/\mu_\beta(\mathbf{0}) \approx \theta_\beta^{-1}$ . On the other hand,  $\mu_\beta(\sigma)/\text{cap}(\sigma, \mathcal{M})$  is the time scale in which the process reaches one of the configurations in  $\mathcal{M}$  starting from  $\sigma$ , a time scale of smaller order than the one in which it jumps between configurations in  $\mathcal{M}$ .

By (7.10), to prove the lemma for  $\xi = \mathbf{0}$ , we just have to show that

$$\lim_{\beta \rightarrow \infty} \mathbb{E}_0 \left[ \frac{1}{\theta_\beta} \int_0^{t\theta_\beta} \mathbf{1}\{\sigma(s) \in \Delta^*\} ds \mathbf{1}\{H_{\{-1, +1\}} \leq t\theta_\beta\} \right] = 0.$$

Since  $\mathbb{P}_0[H_{-1} < H_{+1}] \rightarrow 0$ , we may add in the previous expectation the indicator of the set  $\{H_{+1} < H_{-1}\}$ . Rewrite the integral over the time interval  $[0, t\theta_\beta]$  as the sum of an integral over  $[0, H_{\{-1, +1\}}]$  with one over the time interval  $[H_{\{-1, +1\}}, t\theta_\beta]$ . The expectation of the first one is handled by (7.10). The expectation of the second one, by the strong Markov property, on the set  $\{H_{\{-1, +1\}} \leq t\theta_\beta\} \cap \{H_{+1} < H_{-1}\}$ , is less than or equal to

$$\mathbb{E}_{+1} \left[ \frac{1}{\theta_\beta} \int_0^{t\theta_\beta} \mathbf{1}\{\sigma(s) \in \Delta^*\} ds \right].$$

By the first part of the proof this expectation vanishes as  $\beta \uparrow \infty$ .

It remains to consider the case  $\xi = -1$ . As in the case  $\xi = \mathbf{0}$ , we first estimate the expectation in (7.10), with  $H_{\{0, +1\}}$  instead of  $H_{\{-1, +1\}}$ . Then, we repeat the arguments presented for  $\xi = \mathbf{0}$ , with obvious modifications, to reduce the case  $\xi = -1$  to the case  $\xi = \mathbf{0}$ , which has already been examined.  $\square$

We conclude this section proving the assertion of Remark 2.6. Fix  $\eta \in \mathcal{M}$ ,  $\sigma \in \mathcal{V}_\eta$ ,  $\sigma \neq \eta$ . By (3.3), (3.4) and Assertion 7.B,

$$\mathbb{P}_\sigma[H_{\mathcal{M} \setminus \{\eta\}} < H_\eta] \leq \frac{\text{cap}(\sigma, \mathcal{M} \setminus \{\eta\})}{\text{cap}(\sigma, \eta)} \approx \sum_{\xi \in \mathcal{M} \setminus \{\eta\}} \frac{\text{cap}(\sigma, \xi)}{\text{cap}(\sigma, \eta)} \approx \sum_{\xi \in \mathcal{M} \setminus \{\eta\}} \frac{\text{cap}(\eta, \xi)}{\text{cap}(\sigma, \eta)}. \quad (7.11)$$

By monotonicity of the capacity, the previous expression is bounded by  $2\text{cap}(\eta, \mathcal{M} \setminus \{\eta\})/\text{cap}(\sigma, \eta)$ , which vanishes in view of Assertion 7.A.

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